

# Generators and defining relations for ring of invariants of commuting locally nilpotent derivations or automorphisms

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## Abstract

Let  $A$  be an algebra over a field  $K$  of characteristic zero, let  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$  be *commuting locally nilpotent*  $K$ -derivations such that  $\delta_i(x_j) = \delta_{ij}$ , the Kronecker delta, for some elements  $x_1, \dots, x_s \in A$ . A set of algebra generators for the algebra  $A^\delta := \bigcap_{i=1}^s \ker(\delta_i)$  is found *explicitly* and a set of *defining relations* for the algebra  $A^\delta$  is described. Similarly, given a set  $\sigma_1, \dots, \sigma_s \in \text{Aut}_K(A)$  of *commuting*  $K$ -automorphisms of the algebra  $A$  such that the maps  $\sigma_i - \text{id}_A$  are *locally nilpotent* and  $\sigma_i(x_j) = x_j + \delta_{ij}$ , for some elements  $x_1, \dots, x_s \in A$ . A set of algebra generators for the algebra  $A^\sigma := \{a \in A \mid \sigma_1(a) = \dots = \sigma_s(a) = a\}$  is found *explicitly* and a set of defining relations for the algebra  $A^\sigma$  is described. In general, even for a *finitely generated noncommutative* algebra  $A$  the algebras of invariants  $A^\delta$  and  $A^\sigma$  are *not* finitely generated, *not* (left or right) Noetherian and *does not* satisfy finitely many defining relations (see examples). Though, for a *finitely generated commutative* algebra  $A$  *always* the *opposite* is true. The derivations (or automorphisms) just described appear often in many different situations after (possibly) a localization of the algebra  $A$ .

*Mathematics subject classification 2000: 16W22, 13N15, 14R10, 16S15, 16D30.*

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# 1 Introduction

The following notation will remain **fixed** throughout the paper (if it is not stated otherwise):  $K$  is a field of characteristic zero (not necessarily algebraically closed),  $A$  is an (associative, not necessarily commutative) algebra with 1, module means a left module,  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$  are *commuting locally nilpotent*  $K$ -derivations of the algebra  $A$ ,  $A^\delta := \bigcap_{i=1}^s \ker(\delta_i)$  is *the algebra of invariants* (or *constants*) for the derivations  $\delta := (\delta_1, \dots, \delta_s)$ ;  $\sigma_1, \dots, \sigma_s \in \text{Aut}_K(A)$  are *commuting*  $K$ -automorphisms of the algebra  $A$  such that the maps  $\sigma_i - \text{id}_A$  are *locally nilpotent* (for each  $a \in A$ ,  $(\sigma_i - \text{id}_A)^n(a) = 0$  for all  $n \gg 1$ ),  $A^\sigma := \{a \in A \mid \sigma_1(a) = \dots = \sigma_s(a) = a\}$  is *the algebra of invariants* for the automorphisms  $\sigma := (\sigma_1, \dots, \sigma_s)$ .

Theorem 2.3 describes algebras  $A$  for which there exist a set of commuting locally nilpotent derivations  $\delta_1, \dots, \delta_s$  such that  $\delta_i(x_j) = \delta_{ij}$ , the Kronecker delta, for some elements  $x_1, \dots, x_s \in A$  and all  $1 \leq i, j \leq s$  (the algebras  $A$  are *iterated Ore extensions* of a very special type). Similarly, Theorem 3.2 describes algebras  $A$  for which there exists a set of commuting automorphisms  $\sigma_1, \dots, \sigma_s$  and a set of elements  $x_1, \dots, x_s \in A$  such that the maps  $\sigma_i - \text{id}_A$  are locally nilpotent and  $\sigma_i(x_j) = x_j + \delta_{ij}$  for all  $1 \leq i, j \leq s$  where  $\text{id}_A$  is the identity map of  $A$ . The algebras  $A$  are precisely of the type as in Theorem 2.3 and vice versa. In particular, the problems of finding generators and defining relations for the algebra  $A^\delta$  is the ‘same’ as the identical one for the algebra  $A^\sigma$ . So, we will restrict ourselves mainly to the case of derivations.

*Remark.* Two old open problems, the *Jacobian Conjecture* and the *Dixmier Problem*, are essentially questions about whether certain *commuting* derivations  $\delta_1, \dots, \delta_s$  (of the polynomial algebra or the Weyl algebra, respectively) such that  $\delta_i(x_j) = \delta_{ij}$  for some elements  $x_1, \dots, x_s$  are *locally nilpotent*. In this paper, we will see that this type of derivations is more common than one may expect. Typically, these derivations appear after localization of algebra. In order to study that kind of derivations it is naturally to look at the locally nilpotent case first.

Theorem 2.9 gives *explicitly* a set of algebra generators for the algebra  $A^\delta$  and describes *explicitly* the set of defining relations for the generators. More one can say in the important special cases, Corollary 2.11 ( $A$  is commutative) and Theorem 2.12 (if  $[x_i, A^\delta] \subseteq A^\delta$  for all  $i = 1, \dots, s$ ). Plenty of examples are considered. A connection with rings of differential operators is described (Corollary 2.6). One can produce an example of a finitely generated noncommutative algebra  $A$  such that the algebras  $A^\delta$  and  $A^\sigma$  are *not* finitely generated, *not* left/right Noetherian, and that generators *do not* satisfy finitely many defining relations (see Section 2).

Theorem 4.1 gives *explicitly* a formula for the inverse of an automorphism of the algebra  $A$  that preserves the ring of invariants  $A^\delta$ . As an application, we deduce the inverse formula for an automorphism of the  $n$ ’th Weyl algebra with polynomial coefficients (Theorem 4.2).

Theorem 5.1 describes algebras  $A$  that admit a set of commuting locally nilpotent derivations with left localizable kernels. As an application of Theorem 5.1 of how to find explicitly the integral closure  $\tilde{K}$  of the field  $K$  in the algebra  $A$  is given by Corollary 5.3. Theorem 6.1 gives a construction of simple algebras coming from a set of commuting locally

nilpotent derivations.

Let  $A = A_n \otimes P_m$  be the  $n$ 'th Weyl algebra with polynomial coefficients  $P_m$  and  $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_s := \frac{\partial}{\partial x_s} \in \text{Der}_K(A)$ ,  $s := 2n + m$ , be the formal partial derivatives of the algebra  $A$ , it is a set of commuting locally nilpotent derivations of the algebra  $A$ . Theorem 7.1 establishes a natural isomorphism of the algebra  $\text{End}_K(A)$  and the algebra  $A[[\partial_1, \dots, \partial_s]]$ , and Theorem 7.4 gives a formula (a sort of a 'noncommutative' Taylor formula but for linear maps rather than for series or polynomials) that represents any  $K$ -linear map  $a : A \rightarrow A$  as a formal series  $a = \sum_{\alpha \in \mathbb{N}^s} a_\alpha \partial^\alpha$ ,  $a_\alpha \in A$ . In particular, for any  $\sigma \in \text{Aut}_K(K[x_1, \dots, x_m])$ ,  $\sigma = \sum_{\alpha \in \mathbb{N}^m} \frac{\prod_{i=1}^m (\sigma(x_i) - x_i)^{\alpha_i}}{\alpha!} \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m}$  (Theorem 7.5).

## 2 Generators and defining relations for ring of invariants of commuting locally nilpotent derivations

Let  $A$  be an algebra over a field  $K$  and let  $\delta$  be a  $K$ -derivation of the algebra  $A$ . The kernel  $A^\delta := \ker \delta$  of  $\delta$  is a subalgebra of  $A$ , so-called, the *algebra of invariants* (or *constants*) of  $\delta$ , the union of the vector spaces  $N := N(\delta, A) = \bigcup_{i \geq 0} N_i$  is a positively *filtered* algebra ( $N_i N_j \subseteq N_{i+j}$  for all  $i, j \geq 0$ ) where  $N_i := \ker(\delta^{i+1})$ . Clearly,  $N_0 = A^\delta$  and  $N := \{a \in A \mid \delta^n(a) = 0 \text{ for some natural } n\}$ . A  $K$ -derivation  $\delta$  of the algebra  $A$  is a *locally nilpotent* derivation if for each element  $a \in A$  there exists a natural number  $n = n(a)$  such that  $\delta^n(a) = 0$ . A  $K$ -derivation  $\delta$  is locally nilpotent iff  $A = N(\delta, A)$ .

Given a ring  $R$  and its derivation  $d$ . The *Ore extension*  $R[x; d]$  of  $R$  is a ring freely generated over  $R$  by  $x$  subject to the defining relations:  $xr = rx + d(r)$  for all  $r \in R$ .  $R[x; d] = \bigoplus_{i \geq 0} R x^i = \bigoplus_{i \geq 0} x^i R$  is a left and right free  $R$ -module. Given  $r \in R$ , a derivation  $(\text{ad } r)(s) := [r, s] = rs - sr$  of  $R$  is called an *inner* derivation of  $R$ .

**Lemma 2.1** [1] *Let  $A$  be an algebra over a field  $K$  of characteristic zero and  $\delta$  be a  $K$ -derivation of  $A$  such that  $\delta(x) = 1$  for some  $x \in A$ . Then  $N(\delta, A) = A^\delta[x; d]$  is the Ore extension with coefficients from the algebra  $A^\delta$ , and the derivation  $d$  of the algebra  $A^\delta$  is the restriction of the inner derivation  $\text{ad } x$  of the algebra  $A$  to its subalgebra  $A^\delta$ . For each  $n \geq 0$ ,  $N_n = \bigoplus_{i=0}^n A^\delta x^i = \bigoplus_{i=0}^n x^i A^\delta$ .*

When the algebra  $A$  is commutative the result above is old and well-known.

**Theorem 2.2** [1] *Let  $A$  be an algebra over a field  $K$  of characteristic zero,  $\delta$  be a locally nilpotent  $K$ -derivation of the algebra  $A$  such that  $\delta(x) = 1$  for some  $x \in A$ . Then the  $K$ -linear map  $\phi := \sum_{i \geq 0} (-1)^i \frac{x^i}{i!} \delta^i : A \rightarrow A$  (resp.  $\psi := \sum_{i \geq 0} (-1)^i \delta^i(\cdot) \frac{x^i}{i!} : A \rightarrow A$ ) satisfies the following properties:*

1.  $\phi$  (resp.  $\psi$ ) is a homomorphism of right (resp. left)  $A^\delta$ -modules.
2.  $\phi$  (resp.  $\psi$ ) is a projection onto the algebra  $A^\delta$ :

$$\begin{aligned} \phi : A = A^\delta \oplus xA &\rightarrow A^\delta \oplus xA, \quad a + xb \mapsto a, \quad \text{where } a \in A^\delta, b \in A, \\ \psi : A = A^\delta \oplus Ax &\rightarrow A^\delta \oplus Ax, \quad a + bx \mapsto a, \quad \text{where } a \in A^\delta, b \in A. \end{aligned}$$

In particular,  $\text{im}(\phi) = \text{im}(\psi) = A^\delta$  and  $\phi(y) = y = \psi(y)$  for all  $y \in A^\delta$ .

3.  $\phi(x^i) = \psi(x^i) = 0$ ,  $i \geq 1$ .

4.  $\phi$  and  $\psi$  are algebra homomorphisms provided  $x \in Z(A)$ , the centre of the algebra  $A$ .

The following notation will remain *fixed* till the end of this section (if it is not stated otherwise):  $A$  is an algebra over a field  $K$  of characteristic zero,  $\delta_1, \dots, \delta_s$  are *commuting locally nilpotent*  $K$ -derivations of  $A$ ,  $A^\delta := \cap_{i=1}^s A^{\delta_i}$  is the *algebra of invariants* for the set of derivation  $\delta_1, \dots, \delta_s$  where  $A^{\delta_i} := \ker(\delta_i)$ . The algebra  $A$  is equipped with the filtration  $\{N_i\}_{i \geq 0}$  ( $N_i N_j \subseteq N_{i+j}$ , for  $i, j \geq 0$ ) where  $N_i := \{a \in A \mid \delta^\alpha(a) = 0 \text{ for all } \alpha = (\alpha_i) \in \mathbb{N}^s \text{ with } |\alpha| := \alpha_1 + \dots + \alpha_s > i\}$ , where  $\delta^\alpha := \delta_1^{\alpha_1} \dots \delta_s^{\alpha_s}$ .  $A = \cup_{i \geq 0} N_i$ ,  $N_0 := A^\delta \subset N_1 \subset \dots$ . For  $0 \neq a \in A$ , a unique number  $i$  such that  $a \in N_i \setminus N_{i-1}$  is called the *order* of  $a$ , denoted  $\text{ord}(a)$ . Consider the associated graded algebra  $\text{gr}(A) := \oplus_{i \geq 0} N_i / N_{i-1}$  ( $N_{-1} := 0$ ).

The next theorem is a crucial step in many results that follow.

**Theorem 2.3** *Let  $A$  be an arbitrary algebra over a field  $K$  of characteristic zero. The following statements are equivalent.*

1. *There exist commuting locally nilpotent  $K$ -derivations  $\delta_1, \dots, \delta_s$  of the algebra  $A$  and elements  $x_1, \dots, x_s \in A$  satisfying  $\delta_i(x_j) = \delta_{ij}$ , the Kronecker delta.*
2. *The algebra  $A$  is an iterated Ore extension  $A = B[x_1; d_1] \cdots [x_s; d_s]$  such that  $d_i(B) \subseteq B$  and  $d_i(x_j) \in B$  for all  $1 \leq i, j \leq s$ .*

If, say, the first condition holds, then  $A = A^\delta[x_1; d_1] \cdots [x_s; d_s]$  is an iterated Ore extension of the ring of invariants  $A^\delta := \cap_{i=1}^s A^{\delta_i}$  such that  $d_i := \text{ad}(x_i)$ ,  $[x_i, A^\delta] \subseteq A^\delta$ , and  $[x_i, x_j] \in A^\delta$  for all  $i, j$ . In particular,  $A = \oplus_{\alpha \in \mathbb{N}^s} x^\alpha A^\delta = \oplus_{\alpha \in \mathbb{N}^s} A^\delta x^\alpha$  where  $x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ , and  $A = \cup_{i \geq 0} N_i$  where  $N_i = \oplus_{|\alpha| \leq i} x^\alpha A^\delta = \oplus_{|\alpha| \leq i} A^\delta x^\alpha$  for  $i \geq 0$ .

*Proof.* (1  $\Rightarrow$  2) Applying Lemma 2.1 step by step we have the result ( $A = \oplus_{k \geq 0} A^{\delta_s^k} x_s^k$ , if  $i < s$  then  $x_i = \sum \lambda_{ij} x_s^j$  for some  $\lambda_{ij} \in A^{\delta_s}$ ; now  $0 = \delta_s(x_i) = \sum j \lambda_{ij} x_s^{j-1}$  implies  $x_i \in A^{\delta_s}$ ):

$$A = A^{\delta_s}[x_s; d_s] = (A^{\delta_{s-1}} \cap A^{\delta_s})[x_{s-1}; d_{s-1}][x_s; d_s] = \cdots = A^\delta[x_1; d_1] \cdots [x_s; d_s], \quad (1)$$

where  $d_i := \text{ad}(x_i)$ . For all  $i, j, k$ ,  $\delta_k([x_i, x_j]) = \delta_{ki}[1, x_j] + \delta_{kj}[x_i, 1] = 0$  and  $\delta_k([x_i, A^\delta]) = \delta_{ki}[1, A^\delta] = 0$ , hence all  $[x_i, x_j] \in A^\delta$  and  $[x_i, A^\delta] \subseteq A^\delta$ .

(2  $\Rightarrow$  1) Given an algebra  $A$  as in the second statement. The formal partial derivatives  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_s} \in \text{Der}_B(A)$  satisfy the condition of the first statement.  $\square$

It is obvious that the elements  $x_1, \dots, x_s$  are *not* (left and right) zero divisors in  $A$ . Next, we have many examples of derivations as in Theorem 2.3.

*Example.* Let  $F_n := K\langle x_1, \dots, x_n \rangle$  be a free algebra over the field  $K$ ,  $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n} \in \text{Der}_K(F_n)$  be the formal partial derivatives and  $I$  be an ideal of  $F_n$  which is  $\partial$ -invariant (that is  $\partial_i(I) \subseteq I$  for all  $i$ ). The induced derivations  $\delta_1, \dots, \delta_n \in \text{Der}_K(A)$  where  $A := F_n/I$ ,  $\delta_i(f + I) = \partial_i(f) + I$ ,  $f \in F_n$ , are commuting locally nilpotent derivations of the algebra  $A$  and  $\delta_i(\bar{x}_i) = \delta_{ij}$  for all  $1 \leq i, j \leq n$  where  $\bar{x}_i := x_i + I$ . If the ideal  $I$  is generated by

the commutators  $[x_i, x_j]$ ,  $1 \leq i, j \leq n$ , we have a polynomial algebra  $K[\bar{x}_1, \dots, \bar{x}_n]$  and the derivations  $\delta_1 := \frac{\partial}{\partial \bar{x}_1}, \dots, \delta_n := \frac{\partial}{\partial \bar{x}_n}$ .

**Corollary 2.4** *Let  $A$ ,  $\delta_1, \dots, \delta_s$  and  $x_1, \dots, x_s$  be as in Theorem 2.3, and  $\mathfrak{m}$  be a (two sided) ideal of the algebra  $A^\delta$  which is  $\text{ad}(x_i)$ -invariant for all  $i = 1, \dots, s$ , and  $(\mathfrak{m}) := \text{Am}A$  be the ideal of the algebra  $A$  generated by  $\mathfrak{m}$ , and  $A \rightarrow \bar{A} := A/(\mathfrak{m})$ ,  $a \mapsto \bar{a} := a + (\mathfrak{m})$ . Then  $\bar{A}, \bar{\delta}_1, \dots, \bar{\delta}_s$  and  $\bar{x}_1, \dots, \bar{x}_s$  satisfy the conditions of Theorem 2.3 (where  $\bar{\delta}_i \in \text{Der}_K(\bar{A})$ ,  $\bar{a} \mapsto \bar{\delta}_i(\bar{a})$ ),  $\bar{A}^\delta = \bar{A}^\delta$ , and  $N'_i = \bar{N}_i = \bigoplus_{|\alpha| \leq i} \bar{A}^\delta \bar{x}^\alpha = \bigoplus_{|\alpha| \leq i} \bar{x}^\alpha \bar{A}^\delta$ , for  $i \geq 0$ , where  $\{N_i\}$  and  $\{N'_i\}$  are the filtrations of the algebras  $\bar{A}$  and  $A$  respectively.*

*Proof.* The derivations  $\bar{\delta}_1, \dots, \bar{\delta}_s$  of the algebra  $\bar{A}$  are commuting locally nilpotent derivations such that  $\bar{\delta}_i(\bar{x}_j) = \delta_{ij}\bar{x}_i$ , hence they satisfy the conditions of Theorem 2.3. In particular,  $\bar{A} = \bar{A}^\delta[\bar{x}_1; \bar{d}_1] \cdots [\bar{x}_s; \bar{d}_s] = \bigoplus_{\alpha \in \mathbb{N}^s} \bar{A}^\delta \bar{x}^\alpha = \bigoplus_{\alpha \in \mathbb{N}^s} \bar{x}^\alpha \bar{A}^\delta$  and  $N'_i = \bigoplus_{|\alpha| \leq i} \bar{A}^\delta \bar{x}^\alpha$ ,  $i \geq 0$ . On the other hand,  $A = \bigoplus_{\alpha \in \mathbb{N}^s} A^\delta x^\alpha$  and  $(\mathfrak{m}) = \bigoplus_{\alpha \in \mathbb{N}^s} \mathfrak{m} x^\alpha$ , hence  $\bar{A} = A/(\mathfrak{m}) = \bigoplus_{\alpha \in \mathbb{N}^s} (A^\delta/\mathfrak{m}) \bar{x}^\alpha$ . Comparing the two direct sums for  $\bar{A}$  we must have  $\bar{A}^\delta = \bar{A}^\delta$ , and  $N'_i = \bar{N}_i = \bigoplus_{|\alpha| \leq i} \bar{A}^\delta \bar{x}^\alpha = \bigoplus_{|\alpha| \leq i} \bar{x}^\alpha \bar{A}^\delta$ , for  $i \geq 0$ .  $\square$

The next result is a criterion of when the ring of invariants  $A^\delta$  is left/right Noetherian.

**Corollary 2.5** *Let  $A$ ,  $\delta_1, \dots, \delta_s$  and  $x_1, \dots, x_s$  be as in Theorem 2.3. Then the following statements are equivalent:*

1. *The algebra  $A$  is left (resp. right) Noetherian.*
2. *The algebra  $A^\delta$  is left (resp. right) Noetherian.*
3. *The algebra  $\text{gr}(A)$  is left (resp. right) Noetherian.*

*Proof.* (1  $\Leftrightarrow$  2) It is a well-known fact that if a coefficient ring is left (resp. right) Noetherian then so is an iterated Ore extension, and vice versa (use iteratively an analogue of the Hilbert Basis Theorem for Ore extensions). Now, the first two statements are equivalent by Theorem 2.3.

(2  $\Leftrightarrow$  3) The associated graded algebra  $\text{gr}(A) \simeq A^\delta[\bar{x}_1; \bar{d}_1] \cdots [\bar{x}_s; \bar{d}_s]$  is an iterated Ore extension where  $\bar{x}_i := x_i + N_1/N_0$ ,  $\bar{d}_i = \text{ad}(\bar{x}_i)$ , and  $[\bar{x}_i, \bar{x}_j] = 0$  for all  $i, j$ . Now, repeat the above argument.  $\square$

*Remark.* Using the previous proof one can write down several similar statements for properties that are ‘stable’ under the operations of taking iterated Ore extension and  $\text{gr}(\cdot)$  (eg, ‘being domain’, etc). For a property of ‘being finitely generated algebra’, in general, it is not true that ‘ $A$  is finitely generated  $\Rightarrow A^\delta$  is finitely generated’ (see an example after Theorem 2.9), but for commutative algebras it is the case (Corollary 2.7).

Corollary 2.6 provides natural examples of commuting locally nilpotent derivations (on non-commutative algebras), it also shows that the order filtration  $\{\mathcal{D}(R)_i\}$  on the ring of differential operators is, in fact, the filtration  $\{N_i\}$  for certain commuting locally nilpotent derivations of  $\mathcal{D}(R)$  (this fact may simplify arguments in finding *explicitly* the ring  $\mathcal{D}(R)$  of differential operators in certain cases, see the example below).

Let  $R$  be a commutative finitely generated  $K$ -algebra and  $\mathcal{D}(R) = \cup_{i \geq 0} \mathcal{D}(R)_i$  be its ring of differential operators on the ring  $R$  equipped with the order filtration  $\{\mathcal{D}(R)_i\}$ :  $\mathcal{D}(R)$  is a  $K$ -subalgebra of the algebra  $\text{End}_K(A)$  where  $\mathcal{D}(R)_0 := \text{End}_R(R) \simeq R$  ( $(x \mapsto rx) \leftrightarrow r$ ), and

$$\mathcal{D}(R)_i := \{u \in \text{End}_K(A) : [u, r] \in \mathcal{D}(R)_{i-1} \text{ for all } r \in R\}.$$

**Corollary 2.6** *Let a domain  $R$  be a commutative finitely generated  $K$ -algebra of Krull dimension  $n > 0$ ,  $\mathcal{D}(R) = \cup_{i \geq 0} \mathcal{D}(R)_i$  be the ring of differential operators on  $R$ ,  $x_1, \dots, x_n$  be algebraically independent (over  $K$ ) elements of  $R$ . Then*

1.  $\delta_1 := \text{ad}(x_1), \dots, \delta_n := \text{ad}(x_n)$  is a set of commuting locally nilpotent derivations of the algebra  $\mathcal{D}(R)$ .
2. The order filtration  $\{\mathcal{D}(R)_i\}$  coincides with the filtration  $\{N_i\}$  associated with the derivations  $\delta_1, \dots, \delta_n$ , i.e.  $\mathcal{D}(R)_i = N_i$  for all  $i \geq 0$ . In particular,  $\mathcal{D}(R)^\delta = R$ .

*Proof.* The first statement is obvious. To prove the second statement, note that  $\mathcal{D}(R)_i \subseteq N_i$  for all  $i \geq 0$  which follows directly from the definitions of both filtrations.

Let  $P_n := K[x_1, \dots, x_n]$  and  $Q_n = K(x_1, \dots, x_n)$  be its field of fractions. The field  $Q := \text{Frac}(R)$  of fractions of  $R$  is a *finite separable* field extension of  $Q_n$ . It well-known that one can pick up a nonzero element, say  $r \in R$ , such that the localization  $R_r := R[r^{-1}]$  of  $R$  at the powers of the element  $r$  is a *regular* domain,  $\partial_i(R_r) \subseteq R_r$  for all  $i$ , and  $\text{Der}_K(R_r) = \oplus_{i=1}^n R_r \partial_i$  where  $\partial_i := \frac{\partial}{\partial x_i}$  are the partial derivatives of  $Q_n$  uniquely extended to derivations of the field  $Q$ . Since the algebra  $R_r$  is regular the ring of differential operators  $\mathcal{D}(R_r)$  on the algebra  $R_r$  is generated by the algebra  $R_r$  and  $\text{Der}_K(R_r)$ , hence  $\mathcal{D}(R_r) = \oplus_{\alpha \in \mathbb{N}^n} R_r \partial^\alpha$  and  $\mathcal{D}(R_r)_i = \oplus_{|\alpha| \leq i} R_r \partial^\alpha$ ,  $i \geq 0$ . Comparing these equalities with similar ones from Theorem 2.3:  $\mathcal{D}(R_r) = \oplus_{\alpha \in \mathbb{N}^n} \mathcal{D}(R_r)^\delta \partial^\alpha = \cup_{i \geq 0} N_i(R_r)$  and  $N_i(R_r) = \oplus_{|\alpha| \leq i} \mathcal{D}(R_r)^\delta \partial^\alpha$ ,  $i \geq 0$ , and taking into account the inclusions  $\mathcal{D}(R_r)_i \subseteq N_i(R_r)$  for  $i \geq 0$ , we must have  $\mathcal{D}(R_r)^\delta = R_r$  and  $N_i(R_r) = \mathcal{D}(R_r)_i$ ,  $i \geq 0$ . Since  $\mathcal{D}(R) \subseteq \mathcal{D}(R_r)$  and  $\mathcal{D}(R)_i = \mathcal{D}(R) \cap \mathcal{D}(R_r)_i$  for all  $i \geq 0$ , and  $N_i = \mathcal{D}(R) \cap N_i(R_r)$ ,  $i \geq 0$ , we conclude that  $N_i = \mathcal{D}(R)_i$  for all  $i \geq 0$ .  $\square$

*Example.* As an application of Theorem 2.3 and Corollary 2.6, let us give a short proof of the well-known fact that *the ring of differential operators  $\mathcal{D}(P_n)$  on a polynomial algebra  $P_n := K[x_1, \dots, x_n]$  (so-called, the Weyl algebra) is generated by  $P_n$  and the partial derivatives  $\partial_1, \dots, \partial_n$  of  $P_n$ : the inner derivations  $\delta_1 := -\text{ad}(x_1), \dots, \delta_n := -\text{ad}(x_n)$  of the algebra  $E := \text{End}_K(P_n)$  commute. Let  $N$  be the largest subalgebra of  $E$  on which all the derivations  $\delta_i$  act locally nilpotently (take the sum of all the subalgebras of  $E$  with the last property). Clearly,  $\mathcal{D}(P_n) \subseteq N$  and  $N^\delta = E^\delta = \text{End}_{P_n}(P_n) \simeq P_n$ . By Theorem 2.3,  $N = P_n \langle \partial_1, \dots, \partial_n \rangle \subseteq \mathcal{D}(P_n)$ , hence  $N = \mathcal{D}(P_n)$ , and, by Corollary 2.6,  $\mathcal{D}(P_n)_i = N_i = \oplus_{|\alpha| \leq i} P_n \partial^\alpha$  for all  $i \geq 0$ .*

**Corollary 2.7** *Let  $A$ ,  $\delta_1, \dots, \delta_s$  and  $x_1, \dots, x_s$  be as in Theorem 2.3. Suppose that the elements  $x_1, \dots, x_s$  are central. Then the following statements are equivalent:*

1. The algebra  $A$  is finitely generated.

2. The algebra  $A^\delta$  is finitely generated.

3. The algebra  $\text{gr}(A)$  is finitely generated.

*Proof.* Since the elements  $x_1, \dots, x_s$  are central, by Theorem 2.3,  $A \simeq A^\delta[x_1, \dots, x_s] \simeq \text{gr}(A)$  and  $A^\delta \simeq A/(x_1, \dots, x_s)$ . Now, it is obvious that the statements are equivalent.  $\square$

Till the end of this section we will assume that for the commuting locally nilpotent derivations  $\delta_1, \dots, \delta_s$  of  $A$  there exist elements  $x_1, \dots, x_s \in A$  such that  $\delta_i(x_j) = \delta_{ij}$ , the Kronecker delta.

For each  $i = 1, \dots, s$ , consider the maps from Theorem 2.2,

$$\phi_i := \sum_{k \geq 0} (-1)^k \frac{x_i^k}{k!} \delta_i^k, \quad \psi_i := \sum_{k \geq 0} (-1)^k \delta_i^k(\cdot) \frac{x_i^k}{k!} : A \rightarrow A.$$

The maps  $\phi_i$  and  $\psi_i$  are homomorphisms of *right* and *left*  $A^\delta$ -modules respectively. The maps

$$\phi := \phi_s \phi_{s-1} \cdots \phi_1 : A \rightarrow A, \quad a = \sum_{\alpha \in \mathbb{N}^s} x^\alpha \lambda_\alpha \mapsto \phi(a) = \lambda_0, \quad (2)$$

$$\psi := \psi_1 \psi_2 \cdots \psi_s : A \rightarrow A, \quad a = \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha x^\alpha \mapsto \psi(a) = \lambda_0, \quad (3)$$

are *projections* onto the subalgebra  $A^\delta$  of  $A = A^\delta \oplus (\oplus_{\alpha \in \mathbb{N}^s, \alpha \neq 0} x^\alpha A^\delta)$  and  $A = A^\delta \oplus (\oplus_{\alpha \in \mathbb{N}^s, \alpha \neq 0} A^\delta x^\alpha)$  respectively, they are homomorphisms of right and left  $A^\delta$ -modules respectively.

**Theorem 2.8** *Let  $A$  be as in Theorem 2.3. For any  $a \in A$ ,*

$$a = \sum_{\alpha \in \mathbb{N}^s} x^\alpha \phi\left(\frac{\delta^\alpha}{\alpha!} a\right) = \sum_{\alpha \in \mathbb{N}^s} \psi\left(\frac{\delta^\alpha}{\alpha!} a\right) x^\alpha.$$

*Proof.* If  $a = \sum x^\alpha \lambda_\alpha$ ,  $\lambda_\alpha \in A^\delta$ , then, by (2),  $\phi(\frac{\delta^\alpha}{\alpha!} a) = \lambda_\alpha$ . Similarly, if  $a = \sum \lambda_\alpha x^\alpha$ ,  $\lambda_\alpha \in A^\delta$ , then, by (3),  $\psi(\frac{\delta^\alpha}{\alpha!} a) = \lambda_\alpha$ .  $\square$

So, the identity map  $\text{id} : A \rightarrow A$  has nice presentations

$$\text{id}(\cdot) = \sum_{\alpha \in \mathbb{N}^s} x^\alpha \phi\left(\frac{\delta^\alpha}{\alpha!}(\cdot)\right) = \sum_{\alpha \in \mathbb{N}^s} \psi\left(\frac{\delta^\alpha}{\alpha!}(\cdot)\right) x^\alpha. \quad (4)$$

Clearly,  $\phi(N_i) \subseteq N_i$  and  $\psi(N_i) \subseteq N_i$  for all  $i \geq 0$ . Consider the associated graded algebra  $\text{gr}(A) := \oplus_{i \geq 0} N_i/N_{i-1}$  ( $N_{-1} := 0$ ). So, let  $\text{gr}(\phi), \text{gr}(\psi) : \text{gr}(A) \rightarrow \text{gr}(A)$  be the induced maps (for  $\bar{a} = a + N_{i-1} \in N_i/N_{i-1}$ ,  $\text{gr}(\phi)(\bar{a}) = \phi(a) + N_{i-1}$  and  $\text{gr}(\psi)(\bar{a}) = \psi(a) + N_{i-1}$ ). Let  $D$  be a free multiplicative monoid generated freely by the inner derivations  $\text{ad}(x_1), \dots, \text{ad}(x_s)$  of the algebra  $A$ . There is an obvious action of  $D$  on the algebra  $A$  (and an obvious linear map  $D \rightarrow \text{End}_K(A)$ ). Let  $\{y_i \mid i \in I\}$  be a set of algebra generators for  $A$ . For each  $d \in D$  and  $i \in I$ , let  $z_{d, \alpha, i} := d\phi(\frac{\delta^\alpha}{\alpha!} y_i)$  where  $\alpha! := \alpha_1! \cdots \alpha_s!$  and  $\alpha_i \leq \text{ord}(y_i)$

for all  $i = 1, \dots, s$ . Let  $\text{id}_A$  be the identity map of  $A$ . For each  $d' \in D^* := D \setminus \{\text{id}_A\}$  and  $j = 1, \dots, s$ , let  $x_{d',j} := d'(x_j)$ . By Theorem 2.3, all the elements  $z_{d,\alpha,i}, x_{d',j} \in A^\delta$ . For each  $z_{d,\alpha,i}$  and each  $x_{d',j}$  we attach (noncommutative) variables  $t_{d,\alpha,i}$  and  $X_{d',j}$  respectively. Let  $\mathcal{F} := K\langle t_{d,\alpha,i}, X_{d',j} \mid d \in D, i \in I, \alpha, d' \in D^*, 1 \leq j \leq s \rangle$  be a free associative algebra and  $f(t_{d,\alpha,i}, X_{d',j}) = f(\{t_{d,\alpha,i}, X_{d',j} \mid d \in D, i \in I, \alpha, d' \in D^*, 1 \leq j \leq s\})$  be a typical element of  $\mathcal{F}$  (the symbols in the brackets, i.e.  $t_{d,\alpha,i}, X_{d',j}$ , stand for *all* the non-commutative arguments of the element  $f$ ).

**Theorem 2.9** *The algebra  $A^\delta$  is generated by all the elements  $\{z_{d,\alpha,i}, x_{d',j}\}$  that satisfy the defining relations  $\mathcal{R} = \{f(t_{d,\alpha,i}, X_{d',j}) \in \mathcal{F} \mid f(z_{d,\alpha,i}, x_{d',j}) \in \sum_{i=1}^s x_i A\}$ . Similarly, the algebra  $A^\delta$  is generated by all the elements  $\{z'_{d,\alpha,i} := d\psi(\frac{\delta^\alpha}{\alpha!} y_i), x_{d',j}\}$  that satisfy the defining relations  $\mathcal{R}' = \{f(t_{d,\alpha,i}, X_{d',j}) \in \mathcal{F} \mid f(z'_{d,\alpha,i}, x_{d',j}) \in \sum_{i=1}^s A x_i\}$ .*

*Proof.* Recall that  $A = \bigoplus_{\alpha \in \mathbb{N}^s} x^\alpha A^\delta$ , and so each  $y_i$  is a unique sum  $y_i = \sum x^\alpha y_{\alpha,i}$  where  $y_{\alpha,i} = \phi(\frac{\delta^\alpha}{\alpha!} y_i) \in A^\delta$  (Theorem 2.8) and  $\alpha_i \leq \text{ord}(y_i)$  for all  $i = 1, \dots, s$  (otherwise,  $y_{\alpha,i} = 0$ ). The set  $\{y_i \mid i \in I\}$  is a set of  $K$ -algebra generators for  $A$ , hence so is the set  $\{y_{\alpha,i}, x_1, \dots, x_s \mid i \in I, \alpha\}$  (with obvious restrictions on  $\alpha \in \mathbb{N}^s$  for each  $y_{\alpha,i}$ , that is  $\alpha_i \leq \text{ord}(y_i)$  for all  $i = 1, \dots, s$ ). Since all the  $y_{\alpha,i} \in A^\delta$ ,  $[x_j, x_k] \in A^\delta$ , and  $[x_j, A^\delta] \subseteq A^\delta$  for all  $j, k$ , any element  $a \in A$  can be written as a sum  $a = \sum x^\alpha a_\alpha$  where each coefficient  $a_\alpha$  belongs to the subalgebra, say  $\mathcal{A}$ , of  $A$  generated by all the elements  $\{z_{d,\alpha,i}, x_{d',j}\}$  in the theorem. It follows that  $A^\delta = \phi(A) \subseteq \mathcal{A}$ , the opposite inclusion,  $\mathcal{A} \subseteq A^\delta$ , is obvious. Therefore,  $A^\delta = \mathcal{A}$ .

Since all the elements  $z_{d,\alpha,i}, x_{d',j} \in A^\delta$  and the map  $\phi$  is a projection onto the ring of invariants  $A^\delta$ , an element  $f(t_{d,\alpha,i}, X_{d',j}) \in \mathcal{F}$  is a relation for the set of generators  $\{z_{d,\alpha,i}, x_{d',j}\}$  of the algebra  $A^\delta$ , i.e.  $f(z_{d,\alpha,i}, x_{d',j}) = 0$ , iff  $\phi(f(z_{d,\alpha,i}, x_{d',j})) = 0$  iff  $f(z_{d,\alpha,i}, x_{d',j}) \in \sum_{j=1}^s x_j A$ .

To prove the remaining case, repeat the above arguments making obvious adjustments.

□

**Corollary 2.10** *Let  $\{a_j \mid j \in J\}$  be algebra generators for  $A^\delta$  and  $F_J = K\langle Y_j \mid j \in J \rangle$  be a free algebra. Then  $\mathcal{R} = \{f(Y_j) \in F_J \mid f(a_j) \in \sum_{i=1}^s x_i A\}$  (resp.  $\mathcal{R}' = \{f(Y_j) \in F_J \mid f(a_j) \in \sum_{i=1}^s A x_i\}$ ) are defining relations for the algebra  $A^\delta$ .*

*Proof.* Repeat the arguments as in the proof of Theorem 2.9. □

*Remark.* The proof of Theorem 2.9 shows that the choice of generators there might be not the most economical one if the algebra  $A$  is far from being free (see also Corollary 2.11 and Theorem 2.12). The proof of Theorem 2.9 shows that in order to find generators and defining relations for the algebra  $A^\delta$  one should

1. take algebra generators  $\{y_i\}_{i \in I}$  for the algebra  $A$ ,
2. find the coefficients  $y_{\alpha,i} := \phi(\frac{\delta^\alpha}{\alpha!} y_i) \in A^\delta$  of each element  $y_i = \sum x^\alpha y_{\alpha,i}$ ,
3. choose a basis, say  $\{b_j \mid j \in I'\}$ , of the  $K$ -linear span of all the coefficients  $\{y_{\alpha,i}\}$ ,



4. then the algebra  $A^\delta$  is generated by the elements  $\{d(b_j), x_{d',i} \mid d \in D, d' \in D^*, j \in I', i = 1, \dots, s\}$ ,
5. choose more economically (if possible) an algebra generators say  $\{a_j \mid j \in J\}$  for  $A^\delta$ ,
6. Corollary 2.10 gives the defining relations.

*Example.* Let  $A = F_n = \langle x_1, \dots, x_n \rangle$  be a free associative algebra over  $K$ ,  $\delta_1 := \frac{\partial}{\partial x_1}, \dots, \delta_s := \frac{\partial}{\partial x_s} \in \text{Der}_K(F_n)$  be formal partial derivatives,  $s \leq n$ :  $\delta_i(x_j) = \delta_{ij}$ ,  $1 \leq i \leq s$ ,  $1 \leq j \leq n$ . Then  $\phi(x_1) = \dots = \phi(x_s) = 0$  and  $\phi(x_{s+1}) = x_{s+1}, \dots, \phi(x_n) = x_n$ . By Theorem 2.9,  $A^\delta = K\langle d(x_i), d'(x_j) \mid d \in D, d' \in D^*, i = s+1, \dots, n; j = 1, \dots, s \rangle = K\langle x_i, d'(x_j) \mid d' \in D^*, i = s+1, \dots, n; j = 1, \dots, s \rangle$ , and, by Corollary 2.5, for  $n \geq 2$ , the algebra  $A^\delta$  is *not* left/right Noetherian since  $F_n$  is not. In the special case when  $s = n$ , the algebra  $A^\delta$  is a *free algebra in infinitely many variables*. More precisely, it is generated freely by the elements  $\{(\text{ad } x_1)^{\alpha_1} \dots (\text{ad } x_n)^{\alpha_n}([x_i, x_j]) \mid i < j, (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$  (Prop. 2, [2]). Now, taking any ideal  $\mathfrak{m}$  of the algebra  $A^\delta$  which is *not* finitely generated as an  $A^\delta$ -bimodule and such that the algebra  $A^\delta/\mathfrak{m}$  is not left/right Noetherian, and using Corollary 2.4 one produces an example of an algebra  $\overline{A}^\delta$  which is *not* finitely generated, *not* left/right Noetherian, and *does not* satisfy finitely many defining relations.

**Corollary 2.11** *If, in addition,  $A$  is commutative then the map  $\phi : A \rightarrow A^\delta$  is an algebra epimorphism with  $\ker(\phi) = (x_1, \dots, x_s)$ . In particular,  $A^\delta \simeq A/(x_1, \dots, x_s)$ . Alternatively, the algebra  $A^\delta$  is generated by the elements  $\{\phi(y_i) \mid i \in I\}$  that satisfy the defining relations  $\mathcal{R} = \{f(t_i) \in K[t_i \mid i \in I] \mid f(\phi(y_i)) \in (x_1, \dots, x_s)\}$ .*

*Proof.* Each of the maps  $\phi_i = \psi_i$  is an algebra homomorphism, hence so is their product  $\phi = \psi$ . Now, the result follows from (2) or (3).  $\square$

*Example.* The Weitzenböck derivation  $\delta = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + \dots + x_{n-1} \frac{\partial}{\partial x_n}$  of the polynomial algebra  $P_n := K[x_1, \dots, x_n]$  ( $n \geq 3$ ) is locally nilpotent with  $\delta(x_1) = 0$ . The derivation can be (uniquely) extended to a locally nilpotent derivation of the localization  $P_n[x_1^{-1}] = K[x_1, x_1^{-1}][x_2, \dots, x_n]$  with  $\delta(x) = 1$  where  $x := \frac{x_2}{x_1}$ . By Corollary 2.11, the algebra of invariants  $P_n[x_1^{-1}]^\delta$  is equal to  $K[x_1, x_1^{-1}][\phi(x_3), \dots, \phi(x_n)]$ , the polynomial algebra in  $\phi(x_i)$ ,  $i = 3, \dots, n$ , with coefficients from  $K[x_1, x_1^{-1}]$  (note that  $\phi(x_2) = x_1 \phi(x) = x_1 0 = 0$ ) where

$$\phi(x_i) = \sum_{k=0}^{i-1} (-1)^k \left(\frac{x_2}{x_1}\right)^k \frac{x_{i-k}}{k!}, \quad i = 3, \dots, n.$$

Hence,  $P_n^\delta = P_n \cap P_n[x_1^{-1}]^\delta = P_n \cap K[x_1, x_1^{-1}][\phi(x_3), \dots, \phi(x_n)]$ . Since  $\delta(\sum_{i=1}^s Kx_i) \subseteq \sum_{i=1}^s Kx_i$ , by the Theorem of Weitzenböck,  $P_n^\delta$  is a *finitely generated* algebra. It is an open problem to find explicitly a set of algebra generators for it (it would imply an explicit description of all  $SL_2$ -invariants which is another open problem, in fact, these two problems are equivalent).

**Theorem 2.12** *If  $[x_i, A^\delta] = 0$ ,  $1 \leq i \leq s$ , then the map  $\text{gr}(\phi) : \text{gr}(A) \rightarrow A^\delta$  is an algebra epimorphism with kernel generated by the elements  $\bar{x}_1, \dots, \bar{x}_s$  (where  $\bar{x}_j := x_j + A^\delta \in N_1/N_0$ ). In particular,  $A^\delta \simeq \text{gr}(A)/(\bar{x}_1, \dots, \bar{x}_s)$ . Alternatively, the algebra  $A^\delta$  is generated by the elements  $\{\text{gr}(\phi)(y_i) \mid i \in I\}$  that satisfy the defining relations  $\mathcal{R} = \{f(t_i) \in K\langle t_i \mid i \in I \rangle \mid f(\text{gr}(\phi)(y_i)) \in \sum_{j=1}^s \bar{x}_j \text{gr}(A)\}$ .*

*Proof.* Since  $[x_i, A^\delta] = 0$  and  $[x_i, x_j] \in A^\delta$  for all  $1 \leq i, j \leq s$ , it follows from Theorem 2.3 that  $\text{gr}(A) \simeq A^\delta[\bar{x}_1, \dots, \bar{x}_s]$  is a polynomial algebra over  $A^\delta$  in  $\bar{x}_i := x_i + A^\delta$ ,  $i = 1, \dots, s$ . The induced derivations  $\bar{\delta}_1, \dots, \bar{\delta}_s \in \text{Der}_K(\text{gr}(A))$  of graded degree  $-1$  are commuting locally nilpotent derivations of the algebra  $\text{gr}(A)$  (where  $\bar{\delta}_i : N_j/N_{j-1} \rightarrow N_{j-1}/N_{j-2}$ ,  $a + N_{j-1} \mapsto \delta_i(a) + N_{j-2}$ ) with  $\bar{\delta}_i(\bar{x}_j) = \delta_{ij}$ . Now, we are in the situation of Corollary 2.11. Let  $\bar{\phi}$  be the corresponding map from Corollary 2.11. Clearly,  $\bar{\phi} = \text{gr}(\phi)$ . Now, the result becomes obvious due to Corollary 2.11.  $\square$

**Lemma 2.13** *Let  $A, \delta_1, \dots, \delta_s$ , and  $x_1, \dots, x_s$  be as in Theorem 2.3. If  $A'$  is a  $\delta$ -invariant subalgebra of the algebra  $A$  ( $\delta_i(A') \subseteq A'$  for all  $i$ ) then the restrictions  $\delta'_1 := \delta_1|_{A'}, \dots, \delta'_s := \delta_s|_{A'}$  are commuting locally nilpotent derivations of the algebra  $A'$  and  $N'_i = A' \cap N_i$  for all  $i \geq 0$ , in particular,  $(A')^{\delta'} = A' \cap A^\delta$ , and  $\text{gr}(A') \subseteq \text{gr}(A)$  is a natural inclusion of graded algebras.*

*Proof.* Obvious.  $\square$

*Example.* Given a  $K$ -algebra  $A$  and commuting locally nilpotent derivations  $\delta_1, \dots, \delta_s$  of the algebra  $A$ , and let  $\{N_i\}$  be the corresponding filtration. Let  $Z(A)$  and  $NZD(A)$  be the centre and the set of all the (left and right) non-zero-divisors of  $A$  respectively. Consider the set  $S := A^\delta \cap Z(A) \cap NZD(A)$ . The algebra  $A$  is a subalgebra of the localization  $S^{-1}A$  of the algebra  $A$  at  $S$ , the derivations  $\delta_1, \dots, \delta_s$  can be uniquely extended to derivations of the algebra  $S^{-1}A$ , denoted in the same fashion. These extended derivation  $\delta_1, \dots, \delta_s$  are commuting locally nilpotent derivations of the algebra  $S^{-1}A$ . Suppose that there are elements  $x_1, \dots, x_s \in S^{-1}A$  such that  $\delta_i(x_j) = \delta_{ij}$  for all  $i, j$ . By Lemma 2.13,  $N_i = A \cap \bigoplus_{|\alpha| \leq i} (S^{-1}A)^{\delta} x^\alpha$  for all  $i \geq 0$ .

Fix elements  $a_1, \dots, a_s \in S$ , then the derivations  $\delta'_1 := a_1 \delta_1, \dots, \delta'_s := a_s \delta_s$  of  $A$  are commuting and locally nilpotent with the corresponding filtration  $\{N'_i\}$  on  $A$ . Then  $N'_i = N_i$  for all  $i \geq 0$  where  $\{N_i\}$  is the filtration on  $A$  determined by the derivations  $\delta_1, \dots, \delta_s$ .

More generally, fix elements  $a_1, \dots, a_s, t_1, \dots, t_s \in S$ , then consider derivations  $\delta'_1 := t_1^{-1} a_1 \delta_1, \dots, \delta'_s := t_s^{-1} a_s \delta_s$  of  $S^{-1}A$  which are obviously commuting and locally nilpotent and  $\delta'_i(A') \subseteq A'$  for all  $i$  where  $A' := A[t_1^{-1}, \dots, t_s^{-1}]$ . Let  $A' = \bigcup_{i \geq 0} N'_i$  be the corresponding filtration associated with the derivations  $\delta'_1, \dots, \delta'_s$ . Then, by Lemma 2.13,  $N'_i = A' \cap \bigoplus_{|\alpha| \leq i} (S^{-1}A)^{\delta} x^\alpha$  for all  $i \geq 0$ . Instead of  $A'$  one can take any  $\delta'$ -invariant subalgebra of  $S^{-1}A$ .

### 3 Generators and defining relations for ring of invariants of commuting automorphisms

Let  $A$  be an algebra over a field  $K$ ,  $\sigma \in \text{Aut}_K(A)$ , and  $\delta$  be a  $\sigma$ -derivation of the algebra  $A$ :  $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$  for all  $a, b \in A$ . We will assume that  $\delta\sigma = \sigma\delta$ . Then an induction on  $n$  yields

$$\delta^n(ab) = \sum_{i=0}^n \binom{n}{i} \delta^i(a) \sigma^i \delta^{n-i}(b), \quad n \geq 1. \quad (5)$$

It follows that the  $A^\delta := \ker(\delta)$  is a subalgebra (of constants for  $\delta$ ) of  $A$ , or the ring of invariants, the union  $M := M(\delta, A) = \cup_{i \geq 0} M_i$  of the vector spaces  $M_i := \ker(\delta^{i+1})$  is a positively filtered algebra ( $M_i M_j \subseteq M_{i+j}$  for all  $i, j \geq 0$ ),  $M_0 = A^\delta \subseteq M_1 \subseteq \dots$ . For each  $0 \neq a \in M$ , there exists a unique natural number, say  $d$ , such that  $a \in M_d \setminus M_{d-1}$ . The  $d := \deg(a) = \deg_\delta(a)$  is called the  $\delta$ -degree of the element  $a$ .

*Example.* Given  $\sigma \in \text{Aut}_K(A)$ , then  $\delta := \sigma - 1$  is a  $\sigma$ -derivation of the algebra  $A$  such that  $\delta\sigma = \sigma\delta$ .

Given a vector space  $V$  over the field  $K$ , a  $K$ -linear map  $\varphi : V \rightarrow V$  is called *locally nilpotent* if, for all  $v \in V$ ,  $\varphi^n(v) = 0$  for all  $n \gg 1$ .

Given *commuting*  $K$ -automorphisms  $\sigma_1, \dots, \sigma_s$  of the algebra  $A$  such that the maps  $\sigma_1 - \text{id}_A, \dots, \sigma_s - \text{id}_A$  are *locally nilpotent*. Then the maps  $\sigma_1 - \text{id}_A, \dots, \sigma_s - \text{id}_A$  are *commuting locally nilpotent*  $\sigma_1 - \text{id}_A, \dots, \sigma_s - \text{id}_A$ -derivations respectively, and all the maps  $\sigma_i, \sigma_j - \text{id}_A$  *commute*. The algebra  $A$  has the filtration  $\{M_i\}_{i \geq 0}$  where  $M_i := \{a \in A \mid (\sigma - \text{id}_A)^\alpha(a) = 0 \text{ for all } \alpha \in \mathbb{N}^s \text{ such that } |\alpha| > i\}$  where  $(\sigma - \text{id}_A)^\alpha := \prod_{i=1}^s (\sigma_i - \text{id}_A)^{\alpha_i}$ . Clearly,  $M_0 = A^\sigma := \{a \in A \mid \sigma_1(a) = \dots = \sigma_s(a) = a\}$ , the *ring of  $\sigma$ -invariants*,  $M_0 \subseteq M_1 \subseteq \dots \subseteq M_i \subseteq \dots \subseteq A = \cup_{i \geq 0} M_i$ ,  $M_i M_j \subseteq M_{i+j}$  for all  $i, j \geq 0$  (use (5)).

*Example.* Let  $\sigma_1, \dots, \sigma_s$  be  $K$ -automorphisms of the polynomial algebra  $P_s := K[x_1, \dots, x_s]$  given by the rule  $\sigma_i(x_j) = x_j + \delta_{ij}$ . The automorphisms  $\sigma_i$  commute and all the maps  $\sigma_i - \text{id}_{P_s}$  are locally nilpotent. Then the filtration  $\{M_i\}$  on the polynomial algebra  $P_s$  is the ordinary filtration:  $M_i = \sum_{|\alpha| \leq i} Kx^\alpha$ .

**Lemma 3.1** *Let  $A$ ,  $\delta_1, \dots, \delta_s$ , and  $x_1, \dots, x_s$  be as in Theorem 2.3, and  $\sigma \in \text{Aut}_K(A)$ . Then the automorphism commute with the derivations  $\delta_1, \dots, \delta_s$  iff  $\sigma(A^\delta) = A^\delta$  and  $\sigma(x_i) = x_i + \lambda_i$  for some  $\lambda_i \in A^\delta$ .*

*Proof.* ( $\Rightarrow$ ) If the automorphism  $\sigma$  commutes with derivations  $\delta_i$  then so does its inverse  $\sigma^{-1}$ , and so  $\sigma^{\pm 1}(A^\delta) \subseteq A^\delta$ , hence  $\sigma(A^\delta) = A^\delta$ . By (1),  $\sigma(x_i) = \lambda_i + \sum_{0 \neq \alpha \in \mathbb{N}^s} \lambda_{i,\alpha} x^\alpha$  for some  $\lambda_i, \lambda_{i,\alpha} \in A^\delta$ . Comparing the coefficients of  $x^\alpha$ 's in the system of equations  $\sigma\delta_i(x_j) = \delta_i\sigma(x_j)$ ,  $1 \leq i, j \leq s$ , yields  $\sigma(x_i) = \lambda_i + x_i$  for all  $i$ .

( $\Leftarrow$ ) This implication is obvious.  $\square$

**Theorem 3.2** *Let  $A$  be an arbitrary  $K$ -algebra,  $\sigma_1, \dots, \sigma_s \in \text{Aut}_K(A)$  be automorphisms of the algebra  $A$ . The following statements are equivalent.*

1. The maps  $\sigma_1 - \text{id}_A, \dots, \sigma_s - \text{id}_A$  are commuting locally nilpotent and there exist elements  $x_1, \dots, x_s \in A$  satisfying  $\sigma_i(x_j) = x_j + \delta_{ij}$  (the Kronecker delta) for  $1 \leq i, j \leq s$ .
2.  $\sigma_1 = e^{\delta_1}, \dots, \sigma_s = e^{\delta_s}$  for some commuting locally nilpotent derivations  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$  such that  $\delta_i(x_j) = \delta_{ij}$ ,  $1 \leq i, j \leq s$ , for some elements  $x_1, \dots, x_s \in A$ .

If one of the two equivalent conditions holds then  $\delta_i := \sum_{k \geq 1} (-1)^{k+1} \frac{(\sigma_i - \text{id}_A)^k}{k}$ ,  $A^\delta = A^\sigma := \{a \in A \mid \sigma_1(a) = \dots = \sigma_s(a) = a\}$ , and two sets of  $x$ 's coincide up to adding elements of  $A^\sigma = A^\delta$ . So, one can apply all the previous results in finding generators and defining relations for the algebra  $A^\sigma$  (we leave it to the interested reader to write down the corresponding statements).

*Proof.* It is well-known that if an automorphism  $\sigma \in \text{Aut}_K(A)$  is such that the map  $\sigma - \text{id}_A$  is a locally nilpotent map then  $\sigma = e^\delta = \sum_{k \geq 0} \frac{\delta^k}{k!}$  for a unique locally nilpotent derivation  $\delta = \sum_{k \geq 1} (-1)^{k+1} \frac{(\sigma - \text{id}_A)^k}{k} \in \text{Der}_K(A)$ , and vice versa. It follows that the maps  $\sigma_i - \text{id}_A$  are commuting locally nilpotent iff the derivations  $\delta_i$  are commuting locally nilpotent. Then,  $\sigma_i(x_j) = x_j + \delta_{ij}$  iff  $\delta_i(x_j) = \delta_{ij}$ . It is obvious that  $A^\delta = A^\sigma$ .  $\square$

*Example.* Let  $P_n = K[x_1, \dots, x_n]$  be a polynomial algebra and  $\sigma \in \text{Aut}_K(P_n)$  where  $\sigma(x_i) = x_i + x_{i-1}$ ,  $i = 1, \dots, n$ ,  $x_0 := 1$  (and  $x_i := 0$  for all  $i < 0$ ). Then  $\sigma - \text{id}$  is a locally nilpotent map and  $P_n^\sigma = P_n^\delta$  where  $\delta := \sum_{k \geq 1} (-1)^k \frac{(\sigma - \text{id})^k}{k}$ ,  $\delta(x_1) = 1$ . By Theorem 3.2 and Corollary 2.11, the ring of invariants  $P_n^\sigma = K[\phi(x_2), \dots, \phi(x_n)]$  is a polynomial algebra in  $n - 1$  variables

$$\begin{aligned} \phi(x_i) &= \sum_{k \geq 0} (-1)^k \frac{x_1^k}{k!} \sum_{i_1, \dots, i_k \geq 1} (-1)^{i_1 + \dots + i_k + k} \frac{(\sigma - \text{id})^{i_1 + \dots + i_k}}{i_1 \dots i_k} \\ &= \sum_{k=0}^i \frac{x_1^k}{k!} \sum_{i_1, \dots, i_k \geq 1} (-1)^{i_1 + \dots + i_k} \frac{x_{i-i_1-\dots-i_k}}{i_1 \dots i_k}. \end{aligned}$$

**Corollary 3.3** *Let  $A$  be an arbitrary algebra over the field  $K$ . The following statements are equivalent.*

1. There exist commuting  $K$ -automorphisms  $\sigma_1, \dots, \sigma_s$  of the algebra  $A$  such that the maps  $\sigma_i - \text{id}$  are locally nilpotent and  $\sigma_i(x_j) = x_j + \delta_{ij}$  for some elements  $x_1, \dots, x_s \in A$ .
2. The algebra  $A$  is an iterated Ore extension  $A = B[x_1; d_1] \cdots [x_s; d_s]$  such that  $d_i(B) \subseteq B$  and  $d_i(x_j) \in B$  for all  $1 \leq i, j \leq s$ .

If, say, the first condition holds then  $A = A^\sigma[x_1; d_1] \cdots [x_s; d_s]$  is an iterated Ore extension of the ring  $A^\sigma$  such that  $d_i = \text{ad}(x_i)$ ,  $[x_i, A^\sigma] \subseteq A^\sigma$ , and  $[x_i, x_j] \in A^\sigma$  for all  $i, j$ . In particular,  $A = \bigoplus_{\alpha \in \mathbb{N}^s} x^\alpha A^\sigma = \bigoplus_{\alpha \in \mathbb{N}^s} A^\sigma x^\alpha = \bigcup_{i \geq 0} M_i$  where  $M_i = \bigoplus_{|\alpha| \leq i} A^\sigma x^\alpha$ .

*Proof.* (1  $\Rightarrow$  2) By Theorem 3.2, we have a set  $\delta_1, \dots, \delta_s$  of commuting locally nilpotent derivations of the algebra  $A$  such that  $\delta_i(x_j) = \delta_{ij}$  for all  $i, j$ . By Theorem 2.3, statement 2 holds.

(2  $\Rightarrow$  1) Given the iterated Ore extension as in statement 2. It is easy to check that the  $K$ -automorphisms  $\sigma_1, \dots, \sigma_s \in \text{Aut}_K(A)$  given by the rule  $\sigma_i(x_j) = x_j + \delta_{ij}$  satisfy the conditions of statement 1. The rest follows from Theorem 2.3 and the fact that  $(\sigma - \text{id})^\alpha(x^\beta) = \alpha! \delta_{\alpha, \beta}$  for all  $\alpha, \beta \in \mathbb{N}^s$  such that  $|\alpha| \geq |\beta|$  where  $(\sigma - \text{id})^\alpha := \prod_{i=1}^s (\sigma_i - \text{id})^{\alpha_i}$ .  $\square$

Corollary 3.3 proves that the filtration  $\{M_i\}$  of the algebra  $A$  for the automorphisms  $\sigma_1, \dots, \sigma_s$  coincides with the filtration  $\{N_i\}$  for the derivations  $\delta_1, \dots, \delta_s$  (where  $\sigma_i = e^{\delta_i}$ ), that is  $M_i = N_i$  for all  $i \geq 0$ .

## 4 The inverse map for automorphism that preserve the ring of invariants of derivations

Let  $A$ ,  $\delta_1, \dots, \delta_s$  and  $x_1, \dots, x_s$  be as in Theorem 2.3. Suppose that an automorphism  $\sigma \in \text{Aut}_K(A)$  preserves the ring of invariants  $A^\delta$  ( $\sigma(A^\delta) = A^\delta$ ). Let  $\sigma_\delta := \sigma|_{A^\delta} \in \text{Aut}_K(A^\delta)$ . Suppose we know the inverse  $\sigma_\delta^{-1}$  and the twisted derivations  $\delta'_1 := \sigma \delta_1 \sigma^{-1}, \dots, \delta'_s := \sigma \delta_s \sigma^{-1} \in \text{Der}_K(A)$ , then we can write *explicitly* a formula for the inverse automorphism  $\sigma^{-1}$  of  $\sigma$  (Theorem 4.1 and Theorem 4.2). Since  $A = \bigoplus_{\alpha \in \mathbb{N}^s} A^\delta x^\alpha = \bigoplus_{\alpha \in \mathbb{N}^s} x^\alpha A^\delta$ , the automorphism  $\sigma$  is uniquely determined by its restriction  $\sigma_\delta$  to the ring of invariants  $A^\delta$  and the images of  $x$ 's:

$$x'_1 := \sigma(x_1), \dots, x'_s := \sigma(x_s). \quad (6)$$

The twisted derivations  $\delta'_1 := \sigma \delta_1 \sigma^{-1}, \dots, \delta'_s := \sigma \delta_s \sigma^{-1} \in \text{Der}_K(A)$  is a set of *commuting locally nilpotent* derivations of the algebra  $A$  satisfying  $\delta'_i(x'_j) = \delta_{ij}$ . For each  $i = 1, \dots, s$ , consider the maps

$$\phi'_i := \sum_{k \geq 0} (-1)^k \frac{(x'_i)^k}{k!} (\delta'_i)^k, \quad \psi'_i := \sum_{k \geq 0} (-1)^k (\delta'_i)^k (\cdot) \frac{(x'_i)^k}{k!} : A \rightarrow A$$

which are homomorphisms of right and left  $A^\delta$ -modules respectively. The maps

$$\begin{aligned} \phi_\sigma &:= \phi'_s \phi'_{s-1} \cdots \phi'_1 : A \rightarrow A, \quad a = \sum_{\alpha \in \mathbb{N}^s} x^\alpha a_\alpha \mapsto \phi_\sigma(a) = a_0, \\ \psi_\sigma &:= \psi'_1 \psi'_2 \cdots \psi'_s : A \rightarrow A, \quad a = \sum_{\alpha \in \mathbb{N}^s} a_\alpha x^\alpha \mapsto \psi_\sigma(a) = a_0, \end{aligned}$$

are *projections* onto the subalgebra  $A^\delta$  of  $A = A^\delta \oplus (\bigoplus_{0 \neq \alpha \in \mathbb{N}^s} x^\alpha A^\delta)$  and  $A = A^\delta \oplus (\bigoplus_{0 \neq \alpha \in \mathbb{N}^s} A^\delta x^\alpha)$  respectively, they are homomorphisms of right and left  $A^\delta$ -modules respectively. By Theorem 2.8, for any  $a \in A$ ,

$$a = \sum_{\alpha \in \mathbb{N}^s} (x')^\alpha \phi_\sigma \left( \frac{(\delta')^\alpha}{\alpha!} a \right) = \sum_{\alpha \in \mathbb{N}^s} \psi_\sigma \left( \frac{(\delta')^\alpha}{\alpha!} a \right) x^\alpha.$$

Then applying  $\sigma^{-1}$  we finish the proof of the next theorem.

**Theorem 4.1** *Let  $A$ ,  $\delta_i$ ,  $\delta'_i$ ,  $x_i$ , and  $x'_i$  be as above (i.e. the algebra  $A$  is from Theorem 2.3). For  $a \in A$ ,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} x^\alpha \sigma_\delta^{-1} \phi_\sigma \left( \frac{(\partial')^\alpha}{\alpha!} a \right) = \sum_{\alpha \in \mathbb{N}^s} \sigma_\delta^{-1} \psi_\sigma \left( \frac{(\partial')^\alpha}{\alpha!} a \right) x^\alpha.$$

As an application of Theorem 4.1 we find the inverse map of an automorphism of the Weyl algebra with polynomial coefficients.

The *Weyl* algebra  $A_n = A_n(K)$  is a  $K$ -algebra generated by  $2n$  generators  $x_1, \dots, x_{2n}$  subject to the defining relations:

$$[x_{n+i}, x_j] = \delta_{ij}, \quad [x_i, x_j] = [x_{n+i}, x_{n+j}] = 0 \quad \text{for all } 1 \leq i, j \leq n,$$

where  $\delta_{ij}$  is the Kronecker delta,  $[a, b] := ab - ba$ .

Let  $P_m = K[x_{2n+1}, \dots, x_{2n+m}]$  be a polynomial algebra. The Weyl algebra with polynomial coefficients  $A := A_n \otimes P_m = \bigoplus_{\alpha \in \mathbb{N}^s} Kx^\alpha$  where  $s := 2n + m$ ,  $x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ , the order of the  $x$ 's in the product is *fixed*. The algebra  $A_n \otimes P_m$  admits the finite set of *commuting locally nilpotent* derivations, namely, the 'partial derivatives':  $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_s := \frac{\partial}{\partial x_s}$ . Clearly,  $\partial_i = \text{ad}(x_{n+i})$  and  $\partial_{n+i} = -\text{ad}(x_i)$ ,  $i = 1, \dots, n$ .

Let  $\text{Aut}_K(A_n \otimes P_m)$  be the group of  $K$ -algebra automorphisms of the algebra  $A_n \otimes P_m$ . Given an automorphism  $\sigma \in \text{Aut}_K(A_n \otimes P_m)$ . It is uniquely determined by the elements  $x'_1 := \sigma(x_1), \dots, x'_s := \sigma(x_s)$  of the algebra  $A_n \otimes P_m$ . The centre  $Z := Z(A_n \otimes P_m)$  of the algebra  $A_n \otimes P_m$  is equal to  $P_m$ . Therefore, the restriction  $\sigma|_{P_m} \in \text{Aut}_K(P_m)$ , and so  $\Delta := \det\left(\frac{\partial x'_{2n+i}}{\partial x_{2n+j}}\right) \in K^*$  where  $i, j = 1, \dots, n$ . The corresponding (to the elements  $x'_1, \dots, x'_s$ ) 'partial derivatives' (the set of commuting locally nilpotent derivations of the algebra  $A_n \otimes P_m$ )

$$\partial'_1 := \frac{\partial}{\partial x'_1}, \dots, \partial'_s := \frac{\partial}{\partial x'_s} \quad (7)$$

are equal to

$$\partial'_i := \text{ad}(\sigma(x_{n+i})), \quad \partial'_{n+i} := -\text{ad}(\sigma(x_i)), \quad i = 1, \dots, n, \quad (8)$$

$$\partial'_{2n+j} := \Delta^{-1} \det \begin{pmatrix} \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+1})}{\partial x_{2n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{2n+1}} & \cdots & \frac{\partial}{\partial x_{2n+m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+1}} & \cdots & \frac{\partial \sigma(x_{2n+m})}{\partial x_{2n+m}} \end{pmatrix}, \quad j = 1, \dots, m, \quad (9)$$

where we 'drop'  $\sigma(x_{2n+j})$  in the determinant  $\det\left(\frac{\partial \sigma(x_{2n+k})}{\partial x_{2n+l}}\right)$ . Clearly,  $\partial'_i = \sigma \partial_i \sigma^{-1}$  for  $i = 1, \dots, s$ , and  $A^\partial = K$ , so  $\sigma|_K = \text{id}_K$  is known. Now, one can apply Theorem 4.1 to have the next result.

**Theorem 4.2** [1] (The Inversion Formula) *For each  $\sigma \in \text{Aut}_K(A_n \otimes P_m)$  and  $a \in A_n \otimes P_m$ ,*

$$\sigma^{-1}(a) = \sum_{\alpha \in \mathbb{N}^s} x^\alpha \phi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) = \sum_{\alpha \in \mathbb{N}^s} \psi_\sigma\left(\frac{(\partial')^\alpha}{\alpha!} a\right) x^\alpha,$$

where  $(\partial')^\alpha := (\partial'_1)^{\alpha_1} \cdots (\partial'_s)^{\alpha_s}$  and  $s = 2n + m$ .

## 5 Integral closure and commuting locally nilpotent derivations

In this section, the structure of algebras is described that admit a set of commuting locally nilpotent derivations with left localizable kernels.

For an arbitrary algebra  $A$ , we say that derivations  $\delta_1, \dots, \delta_s$  of the algebra  $A$  have *generic kernels* iff the  $2^s - 1$  sets  $\{\cap_{i \in I} A^{\delta_i} \mid \emptyset \neq I \subseteq \{1, \dots, s\}\}$  are *distinct* (iff the sets  $A^\delta, A_i^\delta := \cap_{j \neq i} A^{\delta_j}, i = 1, \dots, s$  are distinct iff  $A^\delta \neq A_i^\delta$  for  $i = 1, \dots, s$ ). We say that the derivations  $\delta_1, \dots, \delta_s$  of the algebra  $A$  have *left localizable kernels* iff there exists a *left Ore* set  $S$  of the algebra  $A$  such that  $S \subseteq A^\delta \cap A^{\text{reg}}$  and  $S \cap \delta_i(A_i^\delta) \neq \emptyset$  for all  $i = 1, \dots, s$ , where  $A^{\text{reg}}$  is the set of all *regular* elements of the algebra  $A$  (an element  $a \in A$  is *regular* if, by definition, it is not a left and right zero divisor of the algebra  $A$ ). *If the derivations  $\delta_1, \dots, \delta_s$  have left localizable kernels then they have generic kernels:* for each  $i$ , fix  $y_i \in A_i^\delta$  such that  $\delta_i(y_i) \in S$ , then  $y_i \in A_i^\delta \setminus (A^\delta \cup \cup_{j \neq i} A_j^\delta)$ , and so the derivations have generic kernels. Clearly, if there exists elements  $x_1, \dots, x_s \in A$  such that  $\delta_i(x_j) = \delta_{ij}$  then the derivations  $\delta_1, \dots, \delta_s$  have left localizable kernels (but not vice versa): for take  $S = \{1\}$ .

**Theorem 5.1** *Let  $A$  be an (arbitrary) algebra over the field  $K$ . The following statements are equivalent.*

1. *The algebra  $A$  admits a finite set of commuting locally nilpotent derivations, say  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$ , with left localizable kernels.*
2. *There exists a left Ore set  $S$  of the algebra  $A$  such that  $S \subseteq A^{\text{reg}}, S^{-1}A = B[x_1; d_1] \cdots [x_s; d_s]$  is an iterated Ore extension such that  $S \subseteq B, d_i(B) \subseteq B$  and  $d_i(x_j) \in B$  for all  $1 \leq i, j \leq s$ , and the algebra  $A$  is  $\partial_i$ -invariant ( $\partial_i(A) \subseteq A$ ) for all  $1 \leq i \leq s$  where  $\partial_i := \frac{\partial}{\partial x_i} \in \text{Der}_B(S^{-1}A)$  are the formal partial derivatives of the  $B$ -algebra  $S^{-1}A$ .*

*If, say, the first condition holds then there exists a left Ore set  $S \subseteq A^\delta \cap A^{\text{reg}}$  such that  $S^{-1}A = (S^{-1}A)^\delta[x_1; d_1] \cdots [x_s; d_s]$  is an iterated Ore extensions such that  $d_i((S^{-1}A)^\delta) \subseteq (S^{-1}A)^\delta$  and  $d_i(x_j) \in (S^{-1}A)^\delta$  for all  $1 \leq i, j \leq s$ . In particular,  $S^{-1}A = \bigoplus_{\alpha \in \mathbb{N}^s} (S^{-1}A)^\delta x^\alpha = \bigoplus_{\alpha \in \mathbb{N}^s} x^\alpha (S^{-1}A)^\delta$  and  $S^{-1}A = \bigcup_{i \geq 0} N'_i, N'_i = \bigoplus_{|\alpha| \leq i} (S^{-1}A)^\delta x^\alpha = \bigoplus_{|\alpha| \leq i} x^\alpha (S^{-1}A)^\delta, i \geq 0$ . Finally,  $A = \bigcup_{i \geq 0} N_i$  and  $N'_i = S^{-1}N_i$  for all  $i \geq 0$ , in particular,  $S^{-1}(A^\delta) = (S^{-1}A)^\delta$  and  $A^\delta = A \cap (S^{-1}A)^\delta$ .*

*Proof.* (1  $\Rightarrow$  2) The derivations  $\delta_i$  are left localizable, that is, there exists a left Ore set  $S$  of the algebra  $A$  such that  $S \subseteq A^\delta \cap A^{reg}$  and  $S \cap \delta_i(A_i^\delta) \neq \emptyset$  for all  $i = 1, \dots, s$ . So, for each  $i = 1, \dots, s$ , one can pick up an element, say  $y_i \in A_i^\delta$ , such that  $\delta_i(y_i) \in S$ , then for the elements  $x_i := \delta_i(y_i)^{-1}y_i \in S^{-1}A$  we have  $\delta_j(x_i) = \delta_{ij}$ , where the ‘new’ derivation  $\delta_j$  is the unique extension of the ‘old’ derivation  $\delta_j$  to a derivation of the algebra  $S^{-1}A$ . By Theorem 2.3,  $S^{-1}A = B[x_1; d_1] \cdots [x_s; d_s]$  is an iterated Ore extension such that  $B = (S^{-1}A)^\delta$ ,  $d_i(B) \subseteq B$ ,  $d_i(x_j) \in B$  for all  $i, j$ , and  $\delta_i := \frac{\partial}{\partial x_i} \in \text{Der}_B(S^{-1}A)$  are formal partial derivatives over  $B$ . Now, it is obvious that the algebra  $A$  is  $\frac{\partial}{\partial x_i}$ -invariant for all  $i$ .

(2  $\Rightarrow$  1) Suppose that the second statement holds. The derivations  $\partial_1, \dots, \partial_s \in \text{Der}_B(S^{-1}A)$  are commuting locally nilpotent, hence so are their restrictions, say  $\delta_1, \dots, \delta_s$ , to the  $\partial$ -invariant subalgebra  $A$  of  $S^{-1}A$  (the set  $S$  consists of regular elements of the algebra  $A$ , so one can identify the algebra  $A$  with its isomorphic image in  $S^{-1}A$  under the natural monomorphism  $A \rightarrow S^{-1}A$ ,  $a \mapsto \frac{a}{1}$ ). For each  $x_i$ , fix an element  $s_i \in S$  such that  $y_i := s_i x_i \in A$ . Then  $\delta_j(y_i) = s_i \delta_{ij}$  for all  $i, j$ . Since  $S \subseteq A^{reg} \cap A^\delta$  is a left Ore set and  $\delta_i(y_i) = s_i \in \delta_i(A_i^\delta) \cap S$ , the kernels of the derivations  $\delta_1, \dots, \delta_s$  are left localizable. This finishes the proof of the implication.

The rest is a direct consequence of Theorem 2.3 and the fact that  $S \subseteq A^{reg} \cap A^\delta$ .  $\square$

**Corollary 5.2** *Let a  $K$ -algebra  $A$  be a commutative domain. The following statements are equivalent.*

1. *The algebra  $A$  admits a finite set of commuting locally nilpotent derivations, say  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$ , with generic kernels.*
2. *There exists a nonzero element  $t \in A$  such that the localization  $A_t := A[t^{-1}]$  of the algebra  $A$  at the powers of the element  $t$  is a polynomial algebra  $A_t = B[x_1, \dots, x_s]$  such that  $t \in B$  and the algebra  $A$  is  $\partial_i$ -invariant ( $\partial_i(A) \subseteq A$ ) for all  $1 \leq i \leq s$  where  $\partial_i := \frac{\partial}{\partial x_i} \in \text{Der}_B(A_t)$  are the formal partial derivatives of  $A_t$  over  $B$ .*

*Proof.* (1  $\Rightarrow$  2) The derivations  $\delta_i$  have generic kernels, so the algebras  $A^\delta$  and  $A_i^\delta$ ,  $i = 1, \dots, s$ , are distinct. So, for each  $i = 1, \dots, s$ , one can fix an element, say  $y_i \in A_i^\delta$ , such that  $0 \neq \delta_i(y_i) \in A^\delta$ . Then the element  $t := \delta_1(y_1) \cdots \delta_s(y_s) \in A^\delta$  is a nonzero one since the algebra  $A$  is a domain, and the derivations  $\delta_1, \dots, \delta_s$  have (left) localizable generic kernels, for it suffices to take  $S = \{t^i \mid i \geq 0\}$ . Applying Theorem 5.1, we obtain statement 2.

(2  $\Rightarrow$  1) This implication is obvious because of Theorem 5.1.  $\square$

The next result gives explicitly generators and defining relations for the integral closure  $\tilde{K}$  of the field  $K$  in the algebra  $A$ .

**Corollary 5.3** *Let a domain  $A = K\langle y_1, \dots, y_r \rangle$  be an affine commutative  $K$ -algebra of Krull dimension  $s \geq 1$ ,  $\tilde{K}$  be an algebraic closure of the field  $K$  in the algebra  $A$  ( $\tilde{K}$  is a field finite over  $K$ , i.e.  $[\tilde{K} : K] < \infty$ ). The following statements are equivalent.*



1. There exist  $s$  commuting locally nilpotent derivations, say  $\delta_1, \dots, \delta_s \in \text{Der}_K(A)$ , with generic kernels.
2.  $A = \tilde{K}[x_1, \dots, x_s]$  is a polynomial algebra over the field  $\tilde{K}$  in  $s$  variables.
3. There exist derivations  $\delta_1, \dots, \delta_s \in \text{Der}_{\tilde{K}}(A)$  and elements  $x_1, \dots, x_s \in A$  such that  $\delta_i(x_j) = \delta_{ij}$  (the Kronecker delta) for all  $1 \leq i, j \leq s$ .

If, say, the first statement holds and  $\{N_i\}$  be the filtration on  $A$  associated with the derivations  $\delta_1, \dots, \delta_s$  then

(i)  $\dim_K(N_i) = [\tilde{K} : K] \binom{i+s}{s} = \frac{[\tilde{K} : K]}{s!} i^s + \dots$  for all  $i \geq 0$ .

(ii) The map  $\phi : A \rightarrow \tilde{K}$  (from Corollary 2.11) is an algebra epimorphism with kernel  $(x_1, \dots, x_s)$ , i.e.  $\tilde{K} \simeq A/(x_1, \dots, x_s)$ . Alternatively, the field  $\tilde{K}$  is generated over  $K$  by the elements  $\phi(y_1), \dots, \phi(y_r)$  that satisfy the defining relations  $\mathcal{R} = \{f(t_1, \dots, t_r) \in K[t_1, \dots, t_r] \mid f(\phi(y_1), \dots, \phi(y_r)) \in (x_1, \dots, x_s)\}$ .

*Proof.* (1  $\Rightarrow$  3) By Corollary 5.2,  $A_t = A_t^\delta[x_1, \dots, x_s]$  for a nonzero element  $t \in A^\delta$ . Since  $s = \text{Kdim}(A) = \text{Kdim}(A_t)$ , we see that  $A_t^\delta$  is a field (since  $A$  is a domain) which is finite over  $K$ , i.e.  $[A_t^\delta : K] < \infty$ . The element  $t \in A^\delta \subseteq A_t^\delta$  is algebraic, hence  $t^{-1} \in A^\delta$ , and so  $A_t = A$  and  $A = A^\delta[x_1, \dots, x_s]$ . Clearly,  $A^\delta \subseteq \tilde{K}$ . The reverse inclusion is obvious since  $\text{char}(K) = 0$  (if  $u \in \tilde{K}$  then  $f(u) = 0$ ,  $f'(u) := \frac{df}{dx}(u) \neq 0$  for some nonzero polynomial  $f(x) \in K[x]$ , and so  $0 = \delta_i(f(u)) = f'(u)\delta_i(u)$  implies  $\delta_i(u) = 0$  for all  $i$  which means that  $u \in A^\delta$ ), hence  $A^\delta = \tilde{K}$ . It is obvious that  $\delta_i = \frac{\partial}{\partial x_i} \in \text{Der}_{\tilde{K}}(A)$  and  $\delta_i(x_j) = \delta_{ij}$ .

(3  $\Rightarrow$  2) By Theorem 2.3,  $A = A^\delta[x_1, \dots, x_s]$  is a polynomial algebra with coefficients from the algebra of invariants  $A^\delta$ . Repeating the above argument we have  $A^\delta = \tilde{K}$ .

(2  $\Rightarrow$  1) The formal partial derivatives  $\delta_i := \frac{\partial}{\partial x_i} \in \text{Der}_{\tilde{K}}(A)$  are commuting locally nilpotent derivations with generic kernels since  $x_i \in A_i^\delta \setminus (A \cup \bigcup_{j \neq i} A_j^\delta)$ . This finishes the proof of the equivalence of the three statements.

Suppose that the equivalent conditions hold, then  $N_i = \bigoplus_{|\alpha| \leq i} \tilde{K} x^\alpha$ , and so  $\dim_K(N_i) = [\tilde{K} : K] \binom{i+s}{s} = \frac{[\tilde{K} : K]}{s!} i^s + \dots$  for all  $i \geq 0$ , i.e. (i) is proved. The statement (ii) follows from Corollary 2.11.  $\square$

*Remark.* The finite separable field extension  $\tilde{K}/K$  is generated by a single element, say  $x$ , over  $K$ . So, the algebra  $A$  from Corollary 5.3 is generated by  $s+1$  elements  $x, x_1, \dots, x_s$  that satisfy a single defining relations  $f(x) = 0$  for an irreducible polynomial  $f(y) \in K[y]$  of degree  $[\tilde{K} : K]$ .

## 6 A construction of simple algebras

In this section, a construction of simple algebras is given (Theorem 6.1) that comes from a set of commuting locally nilpotent derivations which satisfy the conditions of Theorem 2.3.

**Theorem 6.1** *Let  $A, \delta_1, \dots, \delta_s$  and  $x_1, \dots, x_s$  be as in Theorem 2.3. Given a (two-sided) maximal ideal  $\mathfrak{m}$  of the algebra  $A^\delta$  such that  $[x_i, \mathfrak{m}] \subseteq \mathfrak{m}$  and  $[x_i, Z] \subseteq \mathfrak{m}$  for all  $i = 1, \dots, s$  where  $Z$  is the centre of the factor algebra  $A^\delta/\mathfrak{m}$ . Then the iterated Ore extension  $\mathcal{A} := A/(\mathfrak{m})[t_1, \dots, t_s; \delta_1, \dots, \delta_s]$  of the algebra  $A/(\mathfrak{m})$  is a simple algebra where  $(\mathfrak{m}) := AmA$ , the elements  $t_1, \dots, t_s$  commute, and  $t_i a = at_i + \delta_i(a)$  for all  $a \in A/(\mathfrak{m})$  where  $\delta_i \in \text{Der}_K(A/(\mathfrak{m}))$  is the induced derivation:  $u + (\mathfrak{m}) \mapsto \delta_i(u) + (\mathfrak{m})$ ,  $u \in A$ .*

*Proof.* Using Theorem 2.3 and abusing notation slightly one can write the factor algebra  $A/(\mathfrak{m})$  as the iterated Ore extension  $A^\delta/\mathfrak{m}[x_1; d_1] \cdots [x_s; d_s]$  of the algebra  $A^\delta/\mathfrak{m}$ . So, without loss of generality we can assume that  $\mathfrak{m} = 0$ , that is  $A^\delta$  is a simple algebra. We have to prove that the iterated Ore extension  $\mathcal{A} := A[t_1, \dots, t_s; \delta_1, \dots, \delta_s]$  of the algebra  $A$  is a simple algebra. The algebra  $A^\delta$  is simple, and so its centre  $Z$  is a field that contains the field  $K$ . Let  $I$  be a nonzero ideal of the algebra  $\mathcal{A}$ , we have to show that  $I = \mathcal{A}$ . Recall that  $\mathcal{A} = \bigoplus_{\alpha \in \mathbb{N}^s} At^\alpha$  where  $t^\alpha := t_1^{\alpha_1} \cdots t_s^{\alpha_s}$  and  $A = \bigoplus_{\alpha \in \mathbb{N}^s} A^\delta x^\alpha$ . Fix a nonzero element, say  $a \in I$ . Then  $a = \sum a_\alpha t^\alpha$  for some elements  $a_\alpha \in A$  not all of which are zero. Note that the inner derivation  $\text{ad}(t_i)$  of the algebra  $\mathcal{A}$  is a formal partial derivative  $\frac{\partial}{\partial x_i}$  over  $A^\delta$  of the algebra  $\mathcal{A} = A^\delta \langle x_1, \dots, x_s, t_1, \dots, t_s \rangle$ , that is  $\frac{\partial}{\partial x_i}(A^\delta) = 0$ ,  $\frac{\partial}{\partial x_i}(x_i) = 1$  and  $\frac{\partial}{\partial x_i}(y) = 0$  for all  $y \in \{x_1, \dots, \widehat{x_i}, \dots, x_s, t_1, \dots, t_s\}$  (the hat over a symbol means that it is missed). Note that the ideal  $I$  is  $\frac{\partial}{\partial x_i}$ -invariant for all  $i = 1, \dots, s$ . Applying carefully several times inner derivations of the type  $\text{ad}(t_i) = \frac{\partial}{\partial x_i}$  to the element  $a$  we see that we can assume that all the coefficients  $a_\alpha \in A^\delta$  and not all of which are zero ones. Let  $V \subseteq A^\delta$  be the vector space over the field  $Z$  generated by all the coefficients  $a_\alpha$ . Suppose that a set  $a_\alpha, a_\beta, \dots, a_\gamma$  is a  $Z$ -basis for  $V$ . By the *Density Theorem*, there are elements  $u_1, \dots, u_k, v_1, \dots, v_k \in A^\delta$  such that  $\sum_{i=1}^k u_i a_\alpha v_i = 1$ ,  $\sum_{i=1}^k u_i a_\beta v_i = 0, \dots, \sum_{i=1}^k u_i a_\gamma v_i = 0$ . Applying the map  $A \rightarrow A$ ,  $(\cdot) \mapsto \sum_{i=1}^k u_i(\cdot) v_i$ , to the element  $a$ , we can assume that all the coefficients  $a_\alpha \in Z$  but not all are zero. By the assumption,  $[x_i, Z] = 0$  for all  $i = 1, \dots, s$ . Then applying carefully the inner derivations of the type  $-\text{ad}(x_i)$  to the element  $a$  and taking into account the fact that  $-\text{ad}(x_i)(t_j) = \delta_{ij}$ , we get an element  $0 \neq b \in Z \cap I$ . Hence,  $I = \mathcal{A}$ , as required.  $\square$

*Example.* Let  $A = P_n := K[x_1, \dots, x_n]$ ,  $\delta_1 := \frac{\partial}{\partial x_1}, \dots, \delta_s := \frac{\partial}{\partial x_s} \in \text{Der}_K(P_n)$ ,  $A^\delta = K$ ,  $\mathfrak{m} = 0$ . Then the algebra  $P_n[t_1, \dots, t_s; \delta_1, \dots, \delta_s]$  is the  $n$ 'th Weyl algebra  $A_n$ .

*Example.* Let  $A = F_2 := K \langle x_1, x_2 \rangle$  is the free algebra,  $\delta_1 := \frac{\partial}{\partial x_1}, \delta_2 := \frac{\partial}{\partial x_2} \in \text{Der}_K(F_2)$ , the ideal  $\mathfrak{m}$  of  $F_2^\delta$  is generated by a single element  $[x_2, x_1] - 1$  is  $\text{ad}(x_i)$ -invariant,  $i = 1, 2$ . Then  $F_2^\delta/\mathfrak{m} \simeq K$  and the algebra  $\mathcal{A} = K[x_1][x_2; d_2 := \frac{\partial}{\partial x_1}][t_1, t_2; \delta_1, \delta_2]$  is a simple algebra.

*Example.* Let  $A := F_s = K \langle x_1, \dots, x_s \rangle$ ,  $s \geq 2$ , be a free algebra,  $\delta_1 := \frac{\partial}{\partial x_1}, \dots, \delta_s := \frac{\partial}{\partial x_s}$ . Let  $I$  be an ideal of the algebra  $F_s^\delta$  generated by all the commutators  $[x_i, [x_j, x_k]]$ . Then the factor algebra  $P := F_s^\delta/I$  is a polynomial algebra in  $\binom{s}{2}$  variables  $y_{ij} := [x_i, x_j] + I$  and  $\overline{A} := A/(I) = P[\overline{x}_1; \text{ad } \overline{x}_1] \cdots [\overline{x}_s; \text{ad } \overline{x}_s]$  (see Corollary 2.4). Note that all  $(\text{ad } \overline{x}_i)|_P = 0$  and  $[\overline{x}_i, \overline{x}_j] = y_{ij}$ . Hence, every maximal ideal  $\mathfrak{m}$  of the algebra  $P$  satisfies the conditions of Theorem 6.1 and one can easily see that the algebra  $\mathcal{A}$  is isomorphic to the Weyl algebra  $A_s$  over the field  $L := P/\mathfrak{m}$ . Note that the factor algebra  $\overline{A}/(\mathfrak{m})$  is isomorphic to the tensor product  $A_s \otimes_L P_m$  of the Weyl algebra  $A_s$  and a polynomial algebra  $P_m$  in  $m$  variables such that  $s = 2n + m$  and  $2n$  is the rank of the  $s \times s$  skew symmetric matrix  $(y_{ij} + \mathfrak{m})$  over  $L$ .

*Example.* The same results are true for a *free metabelian algebra*. Let  $J$  be an ideal of the free algebra  $F_s$ ,  $s \geq 2$ , generated by all the double commutators  $[a, [b, c]]$  where  $a, b, c \in F_s$ . The ideal  $J$  is  $\delta$ -invariant. Hence, (by definition) the *free metabelian algebra* is defined as  $R := F_s/J$  and it is isomorphic to the factor algebra of  $\bar{A}$  (from the previous example) by an ideal  $(J')$  generated by an ideal  $J'$  of the polynomial algebra  $P$ , i.e.  $R \simeq P/J'[\bar{x}_1; \text{ad } \bar{x}_1] \cdots [\bar{x}_s; \text{ad } \bar{x}_s]$ . Now, it is obvious (it is a particular case of the previous example) that, for any maximal ideal  $\mathfrak{m}$  of the algebra  $P/J'$ , the algebra  $\mathcal{A}$  is isomorphic to the Weyl algebra  $A_s$  over the field  $L := (P/J')/\mathfrak{m}$ , and the factor algebra  $R/(\mathfrak{m})$  is isomorphic to the tensor product  $A_n \otimes_L P_m$ ,  $s = 2n + m$  as in the example above.

## 7 Linear maps as differential operators

Let  $A := A_n \otimes P_m = \bigoplus_{\alpha \in \mathbb{N}^s} Kx^\alpha$ ,  $s := 2n + m$ , be the  $n$ 'th Weyl algebra with polynomial coefficients  $P_m$ . The set of formal 'partial derivatives'  $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_s := \frac{\partial}{\partial x_s}$  is a set of *commuting locally nilpotent  $K$ -derivations* of the algebra  $A$ . Consider the algebra  $\hat{A} := A[[\partial_1, \dots, \partial_s]] = \bigoplus_{\alpha \in \mathbb{N}^s} A\partial^\alpha$ ,  $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_s^{\alpha_s}$ , of formal (noncommutative) series  $\sum a_\alpha \partial^\alpha$ ,  $a_\alpha \in A$ , with multiplication given by the rule  $\partial_i a = a\partial_i + \partial_i(a)$ ,  $a \in A$ ,  $1 \leq i \leq s$ . The multiplication of series is well-defined since all the derivations commute and are locally nilpotent. Since  $\partial_i \in \text{Der}_K(A) \subseteq \text{End}_K(A)$ , the algebra  $\hat{A}$  is, in fact, a subalgebra of the algebra  $\text{End}_K(A)$  of all  $K$ -linear endomorphisms of the vector space  $A$ . The next theorem shows that they coincide.

**Theorem 7.1**  $\hat{A} = \text{End}_K(A)$ .

*Proof.* The algebra  $A$  has a natural finite dimensional filtration  $\{\mathcal{A}_i := \sum_{|\alpha| \leq i} Kx^\alpha\}_{i \geq 0}$  ( $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$  for all  $i, j \geq 0$ ),  $\dim(\mathcal{A}_i) = \binom{s+i}{s}$ , and  $\partial^\alpha(\mathcal{A}_i) \subseteq \mathcal{A}_{i-|\alpha|}$  for all  $\alpha \in \mathbb{N}^s$  and  $i \geq 0$  (we set  $\mathcal{A}_i := 0$  for negative  $i$ ). We have mentioned in passing that the algebra  $\hat{A}$  is a subalgebra of  $\text{End}_K(A)$ , let us prove this statement, that is *each nonzero series  $a = \sum a_\alpha \partial^\alpha \in \hat{A}$  determines a nonzero linear map*: let  $i := \min\{|\alpha| \mid a_\alpha \neq 0\}$ , fix  $a_\alpha \neq 0$  with  $|\alpha| = i$ , then  $a(x^\alpha) = a_\alpha \alpha! \neq 0$ , as required.

It remains to show that that any linear map  $f \in \text{End}_K(A)$  can be represented by a series  $a = \sum a_\alpha \partial^\alpha \in \hat{A}$ . It means that  $f(x^\beta) = a(x^\beta)$ , for all  $\beta \in \mathbb{N}^s$ . The unknowns coefficients  $a_\alpha \in A$  can be found from this system step by step. Clearly,  $f(1) = a_0$ . Suppose that  $i > 0$  and all the coefficients  $a_\alpha$  with  $|\alpha| < i$  have been found. Then, for each  $\alpha$  such that  $|\alpha| = i$ , the element  $a_\alpha$  can be found (uniquely) from the equation  $f(x^\alpha) = \alpha! a_\alpha + \sum_{|\beta| < i} \partial^\beta(x^\alpha)$ .  $\square$

Now we are ready to give a short direct proof of the fact that  $A_n = \mathcal{D}(P_n)$ .

**Corollary 7.2** *Let  $K$  be a field of characteristic zero. The Weyl algebra  $A_n$  is the ring of differential operators  $\mathcal{D}(P_n)$  with polynomial coefficients.*

*Proof.* Applying Theorem 7.1 to the polynomial algebra  $A = P_n = K[x_1, \dots, x_n]$ , we have  $\text{End}_K(P_n) = \hat{P}_n$ . Let  $\mathbb{N} = \bigoplus_{i=1}^n \mathbb{N}e_i$  where  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ . It

follows from  $[x_i, \partial^\alpha] = \alpha_i \partial^{\alpha - e_i}$  for all  $\alpha$  and  $i$  (and from definition of the ring of differential operators) that the  $j$ 'th term of the *order* filtration of the ring of differential operators  $\mathcal{D}(P_n)$  on  $P_n$  is equal to  $\oplus_{|\alpha| \leq j} P_n \partial^\alpha$ . Hence  $\mathcal{D}(P_n) = \oplus_{\alpha \in \mathbb{N}^n} P_n \partial^\alpha = A_n$ .  $\square$

By definition, the  $\mathbf{m}$ -adic topology on the algebra  $\hat{A}$  is given by the ascending chain of *left ideals* of the algebra  $\hat{A}$  (neighbourhoods of zero)

$$\mathbf{m}^{[0]} := \hat{A} \supset \cdots \supset \mathbf{m}^{[i]} := \sum_{|\alpha| \geq i} \hat{A} \partial^\alpha \supset \cdots \supset \bigcap_{i \geq 0} \mathbf{m}^{[i]} = 0.$$

The algebra  $\hat{A}$  is a *complete* (w.r.t. the  $\mathbf{m}$ -topology) topological algebra. The ‘partial derivatives’ over  $A$ ,  $D_i \in \text{Der}_{A,c}(\hat{A})$ ,  $i = 1, \dots, s$ , are *continuous  $A$ -derivations* of the algebra  $\hat{A}$  such that

$$D_i(\partial_j) = \delta_{ij}, \quad 1 \leq i, j \leq s.$$

**Lemma 7.3** *For each  $i = 1, \dots, s$ , the map  $\Psi_i(\cdot) := \sum_{k \geq 0} (-1)^k D_i^k(\cdot) \frac{\partial_i^k}{k!} : \hat{A} \rightarrow \hat{A}$ , is a homomorphism of left  $\hat{A}^{D_i}$ -modules where  $\hat{A}^{D_i} := \ker_{\hat{A}}(D_i) = A[[\partial_1, \dots, \hat{\partial}_i, \dots, \partial_s]]$ ,  $\hat{A} = \hat{A}^{D_i}[[\partial_i]] = \hat{A}^{D_i} \oplus \hat{A}\partial_i$ , and the following statements hold:*

1. *the map  $\Psi$  is a projection onto the algebra  $\hat{A}^{D_i}$  of  $\hat{A}$ :*

$$\Psi_i : \hat{A} = \hat{A}^{D_i} \oplus \hat{A}\partial_i \rightarrow \hat{A} = \hat{A}^{D_i} \oplus \hat{A}\partial_i, \quad a + b\partial_i \mapsto a, \quad \text{where } a \in \hat{A}^{D_i}, \quad b \in \hat{A}.$$

*In particular,  $\text{im}(\Psi_i) = \hat{A}^{D_i}$  and  $\Psi_i(y) = y$  for all  $y \in \hat{A}^{D_i}$ .*

2.  $\Psi_i(\partial_i^k) = 0$ ,  $k \geq 1$ .

*Proof.* The map  $\Psi_i$  is obviously well-defined since the algebra  $\hat{A}$  is complete and  $\partial_i^k \in \mathbf{m}^{[k]}$ ,  $k \geq 0$ .  $\Psi_i(\partial_i) = \partial_i - \partial_i = 0$ , and  $\Psi_i(y) = y$  for all  $y \in A^\delta$ . For any  $a \in \hat{A}$ ,

$$\begin{aligned} D_i \Psi_i(a) &= D_i(a - D_i(a) \frac{\partial_i}{1!} + D_i^2(a) \frac{\partial_i^2}{2!} - D_i^3(a) \frac{\partial_i^3}{3!} + \cdots) \\ &= D_i(a) - D_i(a) - D_i^2(a) \frac{\partial_i}{1!} + D_i^2(a) \frac{\partial_i}{1!} + D_i^3(a) \frac{\partial_i^2}{2!} - D_i^3(a) \frac{\partial_i^2}{2!} - \cdots \\ &= 0. \end{aligned}$$

Therefore,  $\text{im}(\Psi_i) = \hat{A}^{D_i}$ .

$$\Psi_i(\partial_i^m) = \sum_{k \geq 0} (-1)^k D_i^k(\partial_i^m) \frac{\partial_i^k}{k!} = \left( \sum_{k \geq 0} (-1)^k \frac{m(m-1) \cdots (m-k+1)}{k!} \right) \partial_i^m = (1-1)^m \partial_i^m = 0.$$

Since  $\hat{A} = \hat{A}^{D_i}[[\partial_i]]$ , the map  $\Psi_i$  is an  $\hat{A}^{D_i}$ -endomorphism of the left  $\hat{A}^{D_i}$ -module  $\hat{A}$  and  $\Psi_i(\partial_i^k) = 0$ ,  $k \geq 1$ , the map  $\Psi_i$  is a projection onto the subalgebra  $\hat{A}^{D_i}$  of  $\hat{A}$ .  $\square$

The map

$$\Psi := \Psi_1 \Psi_2 \cdots \Psi_s : \hat{A} \rightarrow \hat{A}, \quad a = \sum_{\alpha \in \mathbb{N}_s} a_\alpha \partial^\alpha \mapsto \Psi(a) = a_0 \quad (10)$$

is a *projection* onto the subalgebra  $A$  of  $\hat{A}$  ( $\hat{A} = A \oplus (\oplus_{0 \neq \alpha \in \mathbb{N}_s} A \partial^\alpha)$ ).

**Theorem 7.4** For any  $a \in \widehat{A} := \text{End}_K(A)$ ,

$$a = \sum_{\alpha \in \mathbb{N}^s} \Psi\left(\frac{D^\alpha}{\alpha!} a\right) \partial^\alpha.$$

*Proof.* If  $a = \sum a_\alpha \partial^\alpha \in \widehat{A}$ ,  $a_\alpha \in A$ , then, by (10),  $\Psi\left(\frac{D^\alpha}{\alpha!} a\right) = a_\alpha$ .  $\square$

So, the identity map  $\text{id}_{\widehat{A}} : \widehat{A} \rightarrow \widehat{A}$  has a nice presentation

$$\text{id}_{\widehat{A}}(\cdot) = \sum_{\alpha \in \mathbb{N}^s} \Psi\left(\frac{D^\alpha}{\alpha!}(\cdot)\right) \partial^\alpha. \quad (11)$$

**Theorem 7.5** For any  $\sigma \in \text{Aut}_K(P_m)$ ,

$$\sigma = \sum_{\alpha \in \mathbb{N}^m} \frac{(\sigma(x) - x)^\alpha}{\alpha!} \partial^\alpha = \text{id}_{P_m} + \sum_{i=1}^m (\sigma(x_i) - x_i) \partial_i + \dots$$

where  $\frac{(\sigma(x) - x)^\alpha}{\alpha!} := \prod_{i=1}^m \frac{(\sigma(x_i) - x_i)^{\alpha_i}}{\alpha_i!}$ .

*Proof.* Let  $\sigma'$  be the sum. Then for any  $a, b \in P_m$ :

$$\begin{aligned} \sigma'(ab) &= \sum_{\alpha \in \mathbb{N}^m} \frac{(\sigma(x) - x)^\alpha}{\alpha!} \partial^\alpha(ab) = \sum_{\alpha \in \mathbb{N}^m} \frac{(\sigma(x) - x)^\alpha}{\alpha!} \sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta} \partial^\beta(a) \partial^\gamma(b) \\ &= \left( \sum_{\beta \in \mathbb{N}^m} \frac{(\sigma(x) - x)^\beta}{\beta!} \partial^\beta(a) \right) \left( \sum_{\gamma \in \mathbb{N}^m} \frac{(\sigma(x) - x)^\gamma}{\gamma!} \partial^\gamma(b) \right) = \sigma'(a) \sigma'(b), \end{aligned}$$

and so  $\sigma' \in \text{Aut}_K(P_m)$ . For each  $i = 1, \dots, s$ ,  $\sigma'(x_i) = x_i + \sigma(x_i) - x_i = \sigma(x_i)$ , hence  $\sigma' = \sigma$ .  $\square$

*Example.* Let  $\sigma \in \text{Aut}_K(P_n)$ ,  $P_n := K[x_1, \dots, x_n]$ ,  $\sigma(x_i) = x_i + \lambda_i$  where  $\lambda := (\lambda_1, \dots, \lambda_n) \in K^n$ . By Theorem 7.5,

$$\sigma = \sum_{\alpha \in \mathbb{N}^s} \frac{\lambda^\alpha \partial^\alpha}{\alpha!} = \prod_{i=1}^n \left( \sum_{k \geq 0} \frac{(\lambda_i \partial_i)^k}{k!} \right) = \prod_{i=1}^n e^{\lambda_i \partial_i} = e^{\sum_{i=1}^n \lambda_i \partial_i}.$$

*Example.* Let  $\sigma_\lambda \in \text{Aut}_K(P_n)$ ,  $P_n := K[x_1, \dots, x_n]$ ,  $\sigma(x_i) = \lambda_i x_i$ ,  $\lambda := (\lambda_1, \dots, \lambda_n) \in K^{*n}$ . By Theorem 7.5,  $\sigma_\lambda = \sum_{\alpha \in \mathbb{N}^s} (\lambda - 1)^\alpha \frac{x^\alpha \partial^\alpha}{\alpha!}$  where  $(\lambda - 1)^\alpha := \prod_{i=1}^n (\lambda_i - 1)^{\alpha_i}$ . Clearly,  $\sigma_\lambda \sigma_\mu = \sigma_{\lambda\mu}$  for all  $\lambda, \mu \in K^{*n}$ .

## References

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